9. Justify that the probability to get a value W at large times behaves as $P(W,t) \simeq \mathrm{e}^{t\pi(W/t)}$, where the function $\pi(w)$ will be related to $\psi(\lambda)$.

Solution: We know that $\hat{p}(v,\lambda,t) = \int dW e^{-\lambda W} p(v, W, t) \sim e^{\psi t - av^2}$ at large times, so that:

$$p(v, W, t) = \int \frac{\mathrm{d}\lambda}{2\pi i} e^{\lambda W} e^{\psi t - av^2} \tag{99}$$

and thus, up to some time-independent prefactor,

$$p(W, t) = \int \frac{\mathrm{d}\lambda}{2\pi i} e^{\lambda W} e^{\psi t} \tag{100}$$

Writing $W = wt$:

$$p(wt, t) = \int \frac{\mathrm{d}\lambda}{2\pi i} e^{t(\lambda w + \psi(\lambda))} \tag{101}$$

and realizing that the λ integral is dominated by λ such that $\lambda w + \psi(\lambda)$ is the largest. The saddle-point approximation then immediately tells us that:

$$p(wt, t) \sim e^{t\pi(w)},\ \pi(w) = \sup_{\lambda}\{\lambda w + \psi(\lambda)\} \tag{102}$$

By solving $\frac{\mathrm{d}}{\mathrm{d}\lambda}[\lambda w + \psi(\lambda)] = 0$, we obtain:

$$\lambda^* = \frac{T}{2w^2} - \frac{1}{4T} \tag{103}$$

So that:

$$\pi(w) = -\frac{(w-T)^2}{4Tw}\Theta(w) \tag{104}$$

10. Qualitatively plot $\pi(w)$ as a function of w. Does this function possess any remarkable symmetry property? Comment on your finding.

Solution:

There is no reason to have any remarkable property for $\pi(w)$, as has been discussed by J. Farago in J. Stat. Phys. **107**, 781 (2002), *Injected Power Fluctuations in Langevin Equation*. We know what the general appearance of π is because it has to be zero at the optimal value of w, which is $\langle w \rangle = -\psi'(0) = T$.

4 The Fokker-Planck Equation

4.1 Heat flux in a two particle system

We consider two coupled, overdamped particles connected to two different thermal baths at temperatures T_1 and T_2, evolving according to

$$\dot{x}_1 = -\mu_1 \partial_{x_1} V(x_1 - x_2) + \sqrt{2\mu_1 T_1}\eta_1,\ \dot{x}_2 = -\mu_2 \partial_{x_2} V(x_1 - x_2) + \sqrt{2\mu_2 T_2}\eta_2 \tag{105}$$

where the Gaussian whites noises η_1 and η_2 are independent,

$$\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t - t'),\ i = 1, 2 \tag{106}$$

The temperatures T_1 and T_2 are not necessarily equal.

1. Let $A(x_1(t), x_2(t))$ be a physical quantity of interest. Explain why

$$\langle A\rangle = \int \mathrm{d}y_1 \mathrm{d}y_2 A(y_1, y_2)\langle \delta(y_1 - x_1(t))\delta(y_2 - x_2(t))\rangle \tag{107}$$

2. Find an evolution equation for the average $p(y_1, y_2, t) = \langle \delta(y_1 - x_1(t))\delta(y_2 - x_2(t))\rangle$.

Solution: This is the standard Fokker-Planck equation for p, which one can derive using Itō's lemma applied to $u(t) = U(x_1(t), x_2(t))$:

$$\dot{u} \overset{0}{=} \partial_{x_1} U \dot{x}_1 + \partial_{x_2} U \dot{x}_2 + \mu_1 T_1 \partial_{x_1}^2 U + \mu_2 T_2 \partial_{x_1}^2 U \tag{108}$$

which converts, onece averaged, into

$$\partial_t p = \mu_1 \left[\partial_{y_1}(\partial_{y_1} V(y_1 - y_2)p) + T_1 \partial_{y_1}^2 p\right] + \mu_2 \left[\partial_{y_2}(\partial_{y_2} V(y_1 - y_2)p) + T_2 \partial_{y_2}^2 p\right] \tag{109}$$

3. The coupling between the two systems is achieved by means of a quadratic potential $V(x_1 - x_2) = \frac{k}{2}(x_1 - x_2)^2$. Show that the dynamics of $z = x_1 - x_2$ is given by

$$\dot{z} = -\mu k z + \sqrt{2\mu T}\xi,\ \langle \xi(t)\xi(t')\rangle = \delta(t - t') \tag{110}$$

where ξ is a Gaussian white noise. Express μ and T in terms of known parameters.

Solution: Using the Langevin equations for x_1 and x_2 we see that

$$\dot{z} = -(\mu_1 + \mu_2)kz + \sqrt{2\mu_1 T_1}\eta_1 - \sqrt{2\mu_2 T_2}\eta_2 \tag{111}$$

hence $\mu = \mu_1 + \mu_2$. The combination $\sqrt{2\mu T}\xi = \sqrt{2\mu_1 T_1}\eta_1 - \sqrt{2\mu_2 T_2}\eta_2$ is indeed a Gaussian white noise, because it is the sum of two such white noises. Its variance is given by

$$2\mu T = 2\mu_1 T_1 + 2\mu_2 T_2 \tag{112}$$

hence $T = \frac{\mu_1 T_1 + \mu_2 T_2}{\mu_1 + \mu_2}$.

4. What is the steady-state distribution $q(z)$ of z? Is it an equilibrium one?

Solution: We recognize an Ornstein-Uhlenbeck process, and since

$$z(t) = z(0)e^{-\mu kt} + \int_0^t dt' \sqrt{2\mu T} e^{-\mu k(t-t')} \xi(t') \tag{113}$$

we see that z has a Gaussian distribution, characterized by its mean and variance:

$$\langle z \rangle = z(0)e^{-\mu kt} \to 0 \text{ as } t \to +\infty \tag{114}$$

and

$$\langle z^2 \rangle \to T/k \tag{115}$$

hence $q(z) = e^{-kx^2/(2T)}/\sqrt{2\pi T/k}$. This is an equilibrium distribution because the dynamics is time-reversible as could be seen from the absence of entropy production. This is an illustration of the fact that when starting with two degrees of freedom that are out of equilibrium, x_1 and x_2, or equivalently $z = x_1 - x_2$ and $Z = \frac{x_1+x_2}{2}$, then losing the information about one of these degrees of freedom (here we have no interest in the position of the center of mass Z), we end up with a process $z(t)$ that is obviously in equilibrium. Hence, the same system, depending on the choice of degrees of freedom that are looked at, can appear to be in, or out of equilibrium.

5. Show $q(y_1 - y_2)$ is a stationary solution of the equation derived in question 2. Is this an acceptable solution?

Solution: Of course, even with $q(z) = e^{-V(z)/T}/Z$ (for an arbitrary potential V coupling the two systems) this would work, but it is not normalizable.

6. How would you define the heat Q_1 received by particle 1 from the thermal bath at T_1?

Solution: That's the work of the force exerted by the thermal bath $\mu_1^{-1}(\dot{x}_1+\sqrt{2\mu_1 T_1}\eta_1)$ on particle 1, that is, in terms of heat received per unit time,

$$\begin{aligned}\frac{dQ_1}{dt} &\overset{1/2}{=} \mu_1^{-1}(\dot{x}_1 + \sqrt{2\mu_1 T_1}\eta_1)\dot{x}_1 \\ &= \partial_{x_1} V \dot{x}_1 \\ &= \partial_{x_1} V(-\mu_1 \partial_{x_1} V + \sqrt{2\mu_1 T_1}\eta_1) \\ &= -\mu_1 k^2 (x_1 - x_2)^2 + k\sqrt{2\mu_1 T_1}(x_1 - x_2)\eta_1\end{aligned} \tag{116}$$

and returning to Itō's discretization we get

$$\frac{dQ_1}{dt} \overset{0}{=} -\mu_1 k^2 (x_1 - x_2)^2 + k\mu_1 T_1 + k\sqrt{2\mu_1 T_1}(x_1 - x_2)\eta_1 \tag{117}$$

7. Show that in the steady-state we have $\langle\frac{\mathrm{d}Q_1}{\mathrm{d}t}\rangle = -\langle\frac{\mathrm{d}Q_2}{\mathrm{d}t}\rangle$, and comment on this result.

Solution: Taking the average in the steady-state of the equation for $\frac{\mathrm{d}Q_1}{\mathrm{d}t}$ leads to

$$\langle\frac{\mathrm{d}Q_1}{\mathrm{d}t}\rangle = -\mu_1 k^2\frac{T}{k} + k\mu_1 T_1 = \frac{\mu_1\mu_2}{\mu_1+\mu_2}(T_1 - T_2) \tag{118}$$

which vanishes when the two baths are at equal temperatures. Of course we have that $\langle\frac{\mathrm{d}Q_1}{\mathrm{d}t}\rangle = -\langle\frac{\mathrm{d}Q_2}{\mathrm{d}t}\rangle$ in the steady-state, because, as we have seen, the whole system, or the variable z, is in equilibrium. But when the temperatures are unequal, there is a heat flow from the hot bath to the cold one made possible by the coupling of the two particles.

4.2 Nonequilibrium driving of a single particle

Consider an overdamped Langevin equation for a particle with position $\mathbf{r}(t)$ evolving according to $\dot{\mathbf{r}} = -\partial_{\mathbf{r}} V + \sqrt{2T}\boldsymbol{\eta}$, where V is a given external potential, and where $\boldsymbol{\eta}$ is a Gaussian white noise with independent components, $\langle\eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t-t')$. We are working in a space of dimension d.

1. Write the Fokker-Planck equation for the probability density p ($p(\mathbf{x},t)\mathrm{d}^d x = \mathrm{Prob}\{\mathbf{x} \leq \mathbf{r}(t) \leq \mathbf{x}+\mathrm{d}\mathbf{x}\}$).

Solution: This is $\partial_t p = \partial_{\mathbf{x}}\cdot(\partial_{\mathbf{x}} Vp)T\partial^2_{\mathbf{x}} p$.

2. Check by whatever means you like that the process is in equilibrium.

Solution: It is certainly not enough to verify that the Boltzmann distribution $p_B(\mathbf{x}) = \mathrm{e}^{-\beta V(\mathbf{x})}/Z$ is a stationary solution of the Fokker-Planck equation. However, if the operator $\mathbb{W}$ defined by $\mathbb{W}f = \partial_{\mathbf{r}}\cdot(\partial_{\mathbf{r}} Vf)T\partial^2_{\mathbf{r}} f$ when acting on some function $f(\mathbf{x})$ verifies $\mathbb{W}^\dagger = p_B\mathbb{W}p_B^{-1}$ then the process is in equilibrium. An alternative is to determine the entropy production, which strictly vanishes when the initial and final distributions are given by p_B:

$$\begin{aligned}\Sigma[\mathbf{r}\text{ from }0\to t_{\mathrm{obs}}] =& \ln p_B(\mathbf{r}(0),0) - \ln p_B(\mathbf{r}(t_{\mathrm{obs}}),t_{\mathrm{obs}}) - \frac{1}{T}\int_0^{t_{\mathrm{obs}}}\mathrm{d}t\partial_{\mathbf{r}} V\cdot\dot{\mathbf{r}}\\ =& -\beta(V(\mathbf{r}(0)) + \beta V(\mathbf{r}(t_{\mathrm{obs}})) - \beta\left[V(\mathbf{r}(t))\right]_0^{t_{\mathrm{obs}}}\\ =& 0\end{aligned} \tag{119}$$

3. As a mathematical game (for now), suppose that we add a force $\delta\mathbf{F} = -A\partial_{\mathbf{r}} V$, where A is a d-dimensional skew symmetric matrix, so that now $\dot{\mathbf{r}} = -(\mathbf{1}+A)\partial_{\mathbf{r}} V + \sqrt{2T}\boldsymbol{\eta}$. Write the new Fokker-Planck equation and find out whether p_B is a stationary solution or not.

Solution: We now have that This is $\partial_t p = \partial_{\mathbf{x}} \cdot ([\mathbf{1}+A)]\partial_{\mathbf{x}} V p) T\partial_{\mathbf{x}}^2 p$. The extra current in the Fokker-Planck equation is $\delta\mathbf{j} = \delta\mathbf{F}p$ and it turns out that this piece is divergenceless:

$$\partial_{\mathbf{x}} \cdot \delta\mathbf{j} = \partial_i (A_{ij}\partial_j V \mathrm{e}^{-\beta V}/Z = A_{ij}(\partial_i\partial_j V)\mathrm{e}^{-\beta V}/Z - \beta A_{ij}(\partial_j V)(\partial_i V)\mathrm{e}^{-\beta V}/Z \quad (120)$$

and both terms vanish (they are both of the form A_{ij} times a symmetric contribution). Hence p_B remains the stationary distribution.

4. Show that with this extra force the stationary-state is not an equilibrium one. What is, in your opinion, the advantage of using such a fictitious and unphysical dynamical evolution, if any?

Solution: The entropy production rate reads, on average,

$$\langle \dot{\Sigma} \rangle = -\beta\langle (A\partial_{\mathbf{r}} V)\cdot\dot{\mathbf{r}}\rangle \quad (121)$$

Using stochastic calculus, we see that

$$\begin{aligned}\langle \dot{\Sigma} \rangle &\overset{1/2}{=} -\beta\langle (A\partial_{\mathbf{r}} V)\cdot(-(\mathbf{1}+A)\partial_{\mathbf{r}} V + \sqrt{2T}\boldsymbol{\eta})\rangle \\ &\overset{0}{=} -\beta\langle (A\partial_{\mathbf{r}} V)\cdot(-(\mathbf{1}+A)\partial_{\mathbf{r}} V)\rangle - \partial_{\mathbf{r}}\cdot(A\partial_{\mathbf{r}} V)\rangle \\ &= \beta\langle (A\partial_{\mathbf{r}} V)^2\rangle \\ &> 0\end{aligned} \quad (122)$$

The interest can be found here [19] or there [12, 13]. This is a means of accelerating stochastic gradient descent in optimization problems.

4.3 Gallavotti-Cohen theorem for a driven particle

A Brownian particle is subjected to both conservative force $-\partial_{\mathbf{r}} V$ and to nonconservative ones $\mathbf{f}$ and it evolves according to $\dot{\mathbf{r}} = -\partial_{\mathbf{r}} V + \mathbf{f} + \sqrt{2T}\boldsymbol{\eta}$ where $\boldsymbol{\eta}$ is a Gaussian white noise with independent components, $\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t-t')$

1. Let W such that $\frac{\mathrm{d}W}{\mathrm{d}t} \overset{1/2}{=} \mathbf{F}\cdot\dot{\mathbf{r}}$, with $\mathbf{F} = -\partial_{\mathbf{r}} V + \mathbf{f}$. What is the physical meaning of W? Rewrite $\frac{\mathrm{d}W}{\mathrm{d}t}$ in Itō form.

Solution: This is the power injected by the total force. Let $\mathbf{F} = \mathbf{f} - \partial_{\mathbf{r}} V$. We have $\frac{\mathrm{d}W}{\mathrm{d}t} \overset{0}{=} \mathbf{F}(\mathbf{F} + \sqrt{2T}\boldsymbol{\eta}) + T\partial_{\mathbf{r}}\mathbf{F}$. Indeed, this can be seen by return to the definition of a Stratonovich discretized expression:

$$W(t + \Delta t) - W(t) = \Delta W = \mathbf{F}(\mathbf{r}(t) + \frac{1}{2}\Delta\mathbf{r}) \cdot (\mathbf{F}\Delta t + \sqrt{2T}\Delta\boldsymbol{\eta}) \tag{123}$$

so that up to order Δt we have

$$\Delta W = \mathbf{F}^2\Delta t + \mathbf{F}(\mathbf{r}(t)) \cdot \sqrt{2T}\Delta\boldsymbol{\eta} + \frac{1}{2}[\partial_i F_j(\mathbf{r}(t))]\Delta r_i \Delta\eta_j + O(\Delta t^{3/2}) \tag{124}$$

and taking the average leads to $\langle\Delta W\rangle/\Delta t = \mathbf{F}^2 + T\partial_{\mathbf{r}}F$ because $\langle\Delta r_i\Delta\eta_j\rangle = \sqrt{2T}\delta_{ij}\Delta t$.

The total work W splits into a conservative contribution and one that is the energy injected by the dissipative force $\mathbf{f}$:

$$W(t_{\text{obs}}) = V(\mathbf{r}(0)) - V(\mathbf{r}(t_{\text{obs}})) + \int_0^{t_{\text{obs}}} \mathrm{d}\mathbf{f} \cdot \dot{\mathbf{r}} \tag{125}$$

2. Write a Fokker-Plank equation for $p(\mathbf{r}, W, t)$. For $z \in \mathbb{C}$, determine an evolution equation for $\hat{p}(\mathbf{r}, z, t) = \int \mathrm{d}W \mathrm{e}^{-zW} p(\mathbf{r}, W, t)$ by identifying the operator $\mathbb{H}(z)$ such that $\partial_t\hat{p} = -\mathbb{H}\hat{p}$.

Solution: We apply Itō's lemma to $\hat{p}(\mathbf{r}, z, t) = \langle\delta(\mathbf{r} - \mathbf{r}(t))\delta(W - W(t))\rangle$ and we get

$$\partial_t p = -\partial_{\mathbf{r}} \cdot (\mathbf{F}p) + T\partial_{\mathbf{r}}^2 p - (\mathbf{F}^2 + T\partial_{\mathbf{r}}\mathbf{F})\partial_W p + T\partial_{\mathbf{r}}^2 p + T\mathbf{F}^2\partial_W^2 p + 2T\partial_W\partial_{\mathbf{r}} \cdot (\mathbf{F}p) \tag{126}$$

An alternative would have been to resort to the infinitesimal moment of $\Delta\mathbf{r}$ and ΔW:

$$\begin{gathered} \frac{\langle\Delta\mathbf{r}\rangle}{\Delta t} = \mathbf{F}, \ \frac{\langle\Delta W\rangle}{\Delta t} = \mathbf{F}^2 + T\partial_{\mathbf{r}} \cdot \mathbf{F} \\ \frac{\langle\Delta\mathbf{r}^2\rangle}{\Delta t} = 2T, \ \frac{\langle\Delta W^2\rangle}{\Delta t} = 2T\mathbf{F}^2, \ \frac{\langle\Delta\mathbf{r}\Delta W\rangle}{\Delta t} = 2T\mathbf{F} \end{gathered} \tag{127}$$

Laplace-Fourier transforming with respect to W amounts to replacing ∂_W with z, hence

$$\partial_t\hat{p} = -\partial_{\mathbf{r}} \cdot (\mathbf{F}\hat{p}) + T\partial_{\mathbf{r}}^2 p - (\mathbf{F}^2 + T\partial_{\mathbf{r}}\mathbf{F})z\hat{p} + T\partial_{\mathbf{r}}^2 p + T\mathbf{F}^2 z^2\hat{p} + 2Tz\partial_{\mathbf{r}} \cdot (\mathbf{F}\hat{p}) \tag{128}$$

3. Verify that for z real we have $\mathbb{H}(z)^\dagger = \mathbb{H}(T^{-1} - z)$.

Solution: We start from $\mathbb{H}(z)\phi$:

$$
\begin{aligned}
-\mathbb{H}(z)\phi =&(2Tz-1)\partial_{\mathbf{r}}\cdot(\mathbf{F}\phi)+T\partial_{\mathbf{r}}^2\phi \\
&+(Tz-1)z\mathbf{F}^2\phi-zT\partial_{\mathbf{r}}\mathbf{F}\phi
\end{aligned}
\tag{129}
$$

and we compute its Hermitian conjugate

$$
\begin{aligned}
-[\mathbb{H}(z)]^{\dagger}\phi =&-(2Tz-1)\mathbf{F}\cdot\partial_{\mathbf{r}}\phi+T\partial_{\mathbf{r}}^2\phi \\
&+(Tz-1)z\mathbf{F}^2\phi-zT\partial_{\mathbf{r}}\mathbf{F}\phi \\
=&-(2Tz-1)\left[\partial_{\mathbf{r}}\cdot(\mathbf{F}\phi)-\partial_{\mathbf{r}}\cdot\mathbf{F}\phi\right]+T\partial_{\mathbf{r}}^2\phi \\
&+(Tz-1)z\mathbf{F}^2\phi-zT\partial_{\mathbf{r}}\mathbf{F}\phi \\
=&-(2Tz-1)\partial_{\mathbf{r}}\cdot(\mathbf{F}\phi)+T\partial_{\mathbf{r}}^2\phi \\
&+(Tz-1)z\mathbf{F}^2\phi+(zT-1)\partial_{\mathbf{r}}\mathbf{F}\phi \\
=&\mathbb{H}(1/T-z)
\end{aligned}
\tag{130}
$$

4. Let $P(W,t)=\int \mathrm{d}^d r p(\mathbf{r},W,t)$ be the probability density to observe W. Show that if $\pi(w)=\lim_{t\to+\infty}\frac{1}{t}\ln P(w\,t,t)$ exists, then it must verify $\pi(w)-\pi(-w)=\beta w$.

Solution: Let $-\psi(z)$ be the lowest eigenvalue of $\mathbb{H}(z)$, then it is also the lowest one for $\mathbb{H}(z)^{\dagger}$ and thus $\psi(z)=\psi(T^{-1}-z)$. At large time we must have that

$$
\mathrm{e}^{-\mathbb{H}(z)t}\simeq \mathrm{e}^{\psi(z)t}\times\dots
\tag{131}
$$

so that $\hat{p}(\mathbf{r},z,t)\propto \mathrm{e}^{\psi(z)t}$ and thus $\int \mathrm{d}^d r\hat{p}(\mathbf{r},z,t)=\langle \mathrm{e}^{-zW}\rangle$ is also proportional to $\mathrm{e}^{\psi(z)t}$:

$$
P(W,t)=\int\frac{\mathrm{d}z}{2\pi i}\mathrm{e}^{zW}\langle \mathrm{e}^{-zW}\rangle\simeq\int\frac{\mathrm{d}z}{2\pi i}\mathrm{e}^{t(zw+\psi(z)}
\tag{132}
$$

for $w=W/t$. We thus find that $\lim_{t\to+\infty}\frac{1}{t}\ln P(W,t)=\pi(w)=\max_z\{zw+\psi(z)\}$. Using that

$$
\pi(-w)=\max_{z'}\{-z'w+\psi(z')\}=\max_{z'}\{-z'w+\psi(1/T-z')\}=\max_{z'}\{(1-z')w+\psi(1/T-z')\}-w/T
\tag{133}
$$

we find that indeed $\pi(w)-\pi(-w)=w/T$.

5. Explain why the second law of thermodynamics amounts to $\langle W\rangle>0$. Is there any truth/meaning to that statement at the level of fluctuating trajectories (without the averaging)?

Solution: The total entropy production in a steady state is

$$\Sigma(t_{\text{obs}}) = \ln \frac{p_{\text{ss}}(\mathbf{r}(0)}{p_{\text{ss}}(\mathbf{r}(t_{\text{obs}})} - T^{-1} V(\mathbf{r}(0)) + T^{-1} V(\mathbf{r}(t_{\text{obs}})) + \frac{1}{T} \int_0^{t_{\text{obs}}} \mathbf{f} \cdot \dot{\mathbf{r}} \mathrm{d}t \tag{134}$$

and so, for long trajectories, $\Sigma \simeq W$ and they have the same asympttic distributions, so that we expect that

$$\lim_{t \to +\infty} \frac{1}{t} \ln \frac{P(\Sigma, t)}{P(-\Sigma, t)} = \Sigma \tag{135}$$

which tells us that the probability of a violation of the second law of thermodynamics (namely, the probability that $\Sigma < 0$) decays exponentially fast in t_{obs} (because Σ is extensive in t_{obs}). Of course, the second law holds on average, $\langle \Sigma \rangle \geq 0$.

5 Thermal ratchets and stochastic engines

5.1 Stochastic Stirling engine

A colloidal particle is in contact with a thermostat at temprature T and controled by optical tweezers imposing an external potential $V(\mathbf{r}) = k\frac{\mathbf{r}^2}{2}$ where $\mathbf{r}$ is the position of the particle. The particle is first equilibrated at temperature T_c within a trap characterized by a stiffness k_c. The stiffness is then varied to $k_h > k_c$ during an isothermal transformation. The next step is to increase the temperature up to T_h at fixed k_h. Then during an isothermal transformation the stiffness is brought back to k_c and the final step is, at fixed stiffness, to cool the system back to T_c.

1. Draw this cycle in a $(k, \langle x^2 \rangle)$ plane assuming that after each transformation each state is an equilibrium one. Why is this machine an engine?

2. Show that the efficiency $\mathcal{E}$ of the engine is expressed, in terms of T_c, T_h and $a = \frac{k_h}{k_c} > 1$ as

$$\mathcal{E} = \frac{(T_h - T_c) \ln a}{T_h - T_c + T_h \ln a} \tag{136}$$

3. Compare with the Carnot efficiency.

4. As implemented in [18], the thermal bath is now replaced with a bath of bacteria. Explain why it is reasonable to consider that the effective dynamics of the passive colloidal particle now becomes an active dynamics, say of the form $\dot{\mathbf{r}} = -\mu k \mathbf{r} + v_0 \mathbf{u}$, with $\mathbf{u}$ a unit vector that decorrelates exponentially fast over a time scale τ inherited from the motion of the bacteria ($\langle \mathbf{u}(t) \cdot \mathbf{u}(t') \rangle = \mathrm{e}^{-\frac{|t-t'|}{\tau}}$).

To make calculations easier, we now work in one space dimension.