Nonequilibrium & Active Systems

Exercises for understanding and training

PCS Master 2 program

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Contents

The sequence of sections follows the chapters of the lectures. Some of the problems will not be presented during the tutorials. Solutions to the problems will be posted at the end of each chapter. Questions on the exercises, before and/or after the live session devoted to solving them, are of course very welcome.

1 Equilibrium Statistical Dynamics

1.1 On and around molecular chaos, Kac's ring

This exercise is a translation of the Kac's ring subsection in the book by Dorfman [9]. It is a toy model illustrating the moelcular chaos hypothesis. In short, it is a toy model helpful in understanding what is at stake if we wanted to prove (instead of postulating) equilibrium statistical mechanics.

We consider a one-dimensional ring with N points, and thus N intervals, with periodic boundary conditions. Each interval between two points contains a ball, which is either black or white. At each time step, all balls shift their position by one unit clockwise. On a fraction of the sites, there exists and operator acting on the color of the ball that hops across the site in question by swapping its color before and after the hop. We are interested in the number of balls of each color after t steps. The number of white balls is $W(t)$ and the number of black balls is $B(t)$. The number of white and black balls right before an operator are denoted by $w(t)$ and $b(t)$. We also introduce $\Delta = B - W$.

1. Write the relationship between $B(t+1)$ and $B(t)$, $w(t)$, $b(t)$. Write a similar equation for $W(t+1)$.

   Solution: We have that $B(t+1) = B(t) + w(t) - b(t)$ and similarly $W(t+1) = W(t) + b(t) - w(t)$.

2. Deduce a relationship between $\Delta(t+1)$, $\Delta(t)$, $b(t)$ and $w(t)$.

   Solution: We directly find that $\Delta(t+1) = \Delta(t) - 2(b(t) - w(t))$.

3. We now assume that the fraction of black or white balls that change color at time t equals the probability μ that an operator is present at a given site (that's the assumption that mimics molecular chaos). Implement this assumption and find $\Delta(t)$ as a function of $\Delta(0)$, μ and t.

   Solution: The assumption consists in asserting that $\frac{b()}{B(t)} = \frac{w(t)}{W(t)} = \mu$, which allows to simplify $b - w = \mu(B - W) = \mu\Delta$ and thus $\dot{\Delta} = (1-2\mu)\Delta$, or $\Delta(t) = (1-2\mu)^t\Delta(0)$. Since $\mu < 1$ we have that $|1-2\mu| < 1$ and $\Delta(t) \to 0$ for $t \gg 1$.

4. Consider the time reversal transformation $t \to -t$ and consider the particular time $t_r = 2N$. Which properties of the system are broken by the explicit solution found in question 3?

Solution: The dynamics being completely reversible, changing t into $-t$ shouldn't change the behavior of Δ (the approximate dynamics on Δ is obviously irreversible). At $t_P = 2N$ the initial state of the system is recovered(because each ball will have passed each marker twice). The expression of Δ is not consistent with this exact property. The time t_P plays the role of a Poincaré recurrence time.

5. We now consider an ensemble of Kac's rings with the same initial ball distribution, though the operators are distributed randomly over the different rings (drawn from the same probability distribution). For a given ring, let $\eta_i(t) = \pm 1$ according to whether a black/white ball lies just before site i, and let $\varepsilon_i = \pm 1$ according to whether there exists, or not, an operator at site i. Express $\Delta(t)$ as a function of the $\eta_i(t)$'s and relate $\eta_{i+1}(t+1)$ to $\eta_i(t)$.

Solution: We have $\Delta(t) = \sum_i \eta_i(t)$ and $\eta_{i+1}(t+1) = \varepsilon_i \eta_i(t)$.

6. Prove that

$$\langle \Delta(t) = \langle \varepsilon_1 ... \varepsilon_t \rangle \Delta(0) \tag{1}$$

where the average brackets $\langle \ldots \rangle$ denote an ensemble average over all rings.

Solution: By iterating $\eta_{i+1}(t+1) = \varepsilon_i \eta_i(t)$ we arrive at $\eta_{i+1}(t+1) = \varepsilon_i \varepsilon_{i-1} \ldots \varepsilon_{i-t} \eta_{i-t}(0)$ so that

$$\Delta(t+1) = \sum_i \varepsilon_i \varepsilon_{i-1} \ldots \varepsilon_{i-t} \eta_{i-t}(0) \tag{2}$$

which does indeed verify $\Delta(2N) = \Delta(0)$. Taking the ensemble average leads to $\langle \Delta(t) \rangle = \langle \varepsilon_1 \ldots \varepsilon_t \rangle \Delta(0)$.

7. Show that for $0 \leq t \leq N$ the ensemble average $\langle \Delta(t) \rangle$ is the same as the $\Delta(t)$ found in question 3. What is happening for $2N \geq t \geq N+1$?

Solution: The probability that a ball passes j markers in t steps is $\binom{t}{j} \mu^j (1-\mu)^{t-j}$, and for j markers $\varepsilon_1 \ldots \varepsilon_t = (-1)^j$ so that for $0 < t < N$ we have

$$\langle \varepsilon_1 \ldots \varepsilon_t \rangle = \sum_{j=0}^{t} \binom{t}{j} \mu^j (1-\mu)^{t-j} (-1)^j = (1-2\mu)^t \tag{3}$$

and we find that $\langle \Delta(t) \rangle = (1-2\mu)^t \Delta(0)$. Just as before. But for $t \geq N+1$ we have to replace t with $2N - t$ in most expressions and we find $\langle \Delta(t) \rangle = (1-2\mu)^{2N-t} \Delta(0)$.

8. What lessons can one draw from the above calculations?

Solution: This is an intructive parabola on the Boltzmann equation!

2 The Langevin Equation

2.1 Recipe for a Gaussian white noise

This is an exercise from Van Kampen's book [21]. The goal of this exercise is to show that a Gaussian white noise can be seen as the limiting process of a family of continuous time random signals. We consider a sequence of times t_i randomly distributed with a density ν along the time axis. Let the c_i be identically distributed independent random numbers, with zero average and finite moments, associated to each t_i. Finally, let $\psi(t)$ be a nonnegative function such that $\int \mathrm{d}x\psi(x) = 1$. We consider the random process

$$x(t) = \frac{1}{\sqrt{\nu}} \sum_i c_i \frac{1}{\tau} \psi(\frac{t - t_i}{\tau}) \tag{4}$$

Prove that in the

$$\tau \to 0, \qquad \nu \to \infty \tag{5}$$

limit, the process $x(t)$ is a Gaussian white noise. It may be useful to first determine the generating functional of $x(t)$.

Solution: Let's consider the generating functional

$$Z[h] = \langle \mathrm{e}^{-\int_{-\infty}^{+\infty} h(t)x(t)\mathrm{d}t} \rangle \tag{6}$$

The angular brackets denote a multiple average over: the number n of points t_i in the interval $[0, t_{\mathrm{obs}}]$ (where t_{obs} is, for now, some arbitrary observation time); the uniformly distributed value of each t_i in $[0, t_{\mathrm{obs}}]$; the distribution of the c_i's. The number of points n is drawn from a Poisson distribution $p_n = \mathrm{e}^{-\nu t_{\mathrm{obs}}} \frac{(\nu t_{\mathrm{obs}})^n}{n!}$, and the t_i's have measure $\frac{\mathrm{d}t_i}{t_{\mathrm{obs}}}$, hence

$$Z[h] = \sum_{n\geq 0} \mathrm{e}^{-\nu t_{\mathrm{obs}}} \frac{(\nu t_{\mathrm{obs}})^n}{n!} \int_0^{t_{\mathrm{obs}}} \frac{\mathrm{d}t_1}{t_{\mathrm{obs}}} \dots \frac{\mathrm{d}t_n}{t_{\mathrm{obs}}} \langle \mathrm{e}^{-\int_{-\infty}^{+\infty} h(t) \frac{1}{\sqrt{\nu}} \sum_i c_i \frac{1}{\tau} \psi(\frac{t-t_i}{\tau}) \mathrm{d}t} \rangle \tag{7}$$

where the remaining brackets refer to an average over the c_i's. Using the independence of the t_i's and of the c_i's, we have

$$\begin{aligned}
Z[h] =& \sum_{n\geq 0} \mathrm{e}^{-\nu t_{\mathrm{obs}}} \frac{(\nu t_{\mathrm{obs}})^n}{n!} \left[\int_0^{t_{\mathrm{obs}}} \frac{\mathrm{d}t'}{t_{\mathrm{obs}}} \langle \mathrm{e}^{-\int_{-\infty}^{+\infty} h(t) \frac{1}{\sqrt{\nu}} c \frac{1}{\tau} \psi(\frac{t-t'}{\tau}) \mathrm{d}t} \rangle \right]^n \\
=& \mathrm{e}^{-\nu t_{\mathrm{obs}}} \exp\left[\nu \int_0^{t_{\mathrm{obs}}} \mathrm{d}t' \langle \mathrm{e}^{-\int_{-\infty}^{+\infty} h(t) \frac{1}{\sqrt{\nu}} c \frac{1}{\tau} \psi(\frac{t-t'}{\tau}) \mathrm{d}t} \rangle \right] \\
=& \exp\left[\nu \int_0^{t_{\mathrm{obs}}} \mathrm{d}t' \left(\langle \mathrm{e}^{-\int_{-\infty}^{+\infty} h(t) \frac{1}{\sqrt{\nu}} c \frac{1}{\tau} \psi(\frac{t-t'}{\tau}) \mathrm{d}t} \rangle - 1 \right) \right]
\end{aligned} \tag{8}$$

Finally, we send t_{obs} to $+\infty$. Then, we change variable in the integral appearing in the exponential, $u = \frac{t-t'}{\tau}$, so that

$$
\begin{aligned}
Z[h] &= \exp\left[\nu \int_0^{+\infty} \mathrm{d}t' \left(\langle \mathrm{e}^{-\int_{-\infty}^{+\infty} h(t'+\tau u)\frac{1}{\sqrt{\nu}} c\,\psi(u)\mathrm{d}t}\rangle - 1\right)\right] \\
Z[h] &\overset{\tau\to 0}{=} \exp\left[\nu \int_0^{+\infty} \mathrm{d}t' \left(\langle \mathrm{e}^{-h(t')\frac{1}{\sqrt{\nu}} c \int \mathrm{d}u\psi(u)}\rangle - 1\right)\right]
\end{aligned}
\tag{9}
$$

Expanding the above for large ν leaves us with

$$
Z[h] = \exp\left[\frac{1}{2}\langle c^2\rangle \int_0^{+\infty} \mathrm{d}t' h(t')^2\right] \tag{10}
$$

where we have also used that $\langle c\rangle = 0$. This is the generating functional of a Gaussian white noise with correlations $\langle x(t)x(t')\rangle = \langle c^2\rangle\delta(t-t')$.

2.2 A white, but non-Gaussian, noise

Consider the time interval $[0, t_{\text{obs}}]$ and the let $\eta(t) = \sum_{i=1}^{n} \ell_i \delta(t - t_i)$ be defined over that interval. Here n is a random integer drawn from a Poisson distribution with average νt_{obs} ($\nu > 0$ is a parameter). The ℓ_i's are random numbers drawn from a probability density $\pi(\ell)$ (to make things simpler, we'll assume $\langle \ell_i\rangle = 0$), and the t_i's are uniform random times in the $[0, t_{\text{obs}}]$ interval.

1. Show that the generating functional $G[h] = \langle \exp\left(\int_0^{t_{\text{obs}}} \mathrm{d}t h(t)\eta(t)\right)\rangle$ of the noise has the expression

$$
G[h] = \exp\left[\nu \int_0^{t_{\text{obs}}} \mathrm{d}t \left(\langle \mathrm{e}^{h(t)\ell}\rangle_\pi - 1\right)\right] \tag{11}
$$

where $\langle \ldots\rangle_\pi = \int \mathrm{d}\ell\pi(\ell)\ldots$.

Solution: By definition

$$
\begin{aligned}
G[h] &= \sum_{n=0}^{+\infty} \mathrm{e}^{-\nu t_{\text{obs}}} \frac{(\nu t_{\text{obs}})^n}{n!} \int_0^{t_{\text{obs}}} \frac{\mathrm{d}t_1}{t_{\text{obs}}} \cdots \frac{\mathrm{d}r_n}{t_{\text{obs}}} \langle \mathrm{e}^{\sum_i \ell_i h(t_i)}\rangle_\pi \\
&= \sum_{n=0}^{+\infty} \mathrm{e}^{-\nu t_{\text{obs}}} \frac{(\nu t_{\text{obs}})^n}{n!} \int_0^{t_{\text{obs}}} \frac{\mathrm{d}t_1}{t_{\text{obs}}} \cdots \frac{\mathrm{d}r_n}{t_{\text{obs}}} \prod_{i=1}^{n} \langle \mathrm{e}^{\ell_i h(t_i)}\rangle_\pi \\
&= \sum_{n=0}^{+\infty} \mathrm{e}^{-\nu t_{\text{obs}}} \frac{(\nu t_{\text{obs}})^n}{n!} \left[\int_0^{t_{\text{obs}}} \frac{\mathrm{d}t}{t_{\text{obs}}} \langle \mathrm{e}^{\ell h(t)}\rangle_\pi\right]^n \\
&= \exp\left[\nu \int_0^{t_{\text{obs}}} \mathrm{d}t \left(\langle \mathrm{e}^{h(t)\ell}\rangle_\pi - 1\right)\right]
\end{aligned}
\tag{12}
$$

2. Show that $\langle \eta(t_1)\dots\eta(t_n)\rangle_c = \nu\langle \ell^n\rangle_\pi \delta(t_1-t_2)\dots\delta(t_{n-1}-t_n)$.

Solution: This can obtained by differentiating the generating functional of the cumulants, namely $\ln G[h] = \nu\int_0^{t_{\text{obs}}} \mathrm{d}t\left(\langle \mathrm{e}^{h(t)\ell}\rangle_\pi - 1\right)$, with respect to $h(t_1),\dots,h(t_n)$:

$$\frac{\delta \ln G}{\delta h(t_1)} = \nu\langle \ell \mathrm{e}^{h(t_1)\ell}\rangle_\pi \tag{13}$$

and then sending $h\to 0$. Or it can also be obtained by a direct evaluation:

$$\begin{aligned}
\langle \eta(t_1)\eta(t_2)\eta(t_3)\rangle =& \langle \sum_{i,j,k} \ell_i\ell_j\ell_k\delta(t_1-t_i)\delta(t_2-t_j)\delta(t_3-t_k)\rangle \\
=& \langle \sum_{i,j,k} \underbrace{\langle \ell_i\ell_j\ell_k\rangle_\pi}_{\langle \ell^3\rangle_\pi\delta_{ij}\delta_{ik}} \delta(t_1-t_i)\delta(t_2-t_j)\delta(t_3-t_k)\rangle \\
=& \langle \sum_i \langle \ell^3\rangle_\pi \delta(t_1-t_i)\delta(t_2-t_i)\delta(t_3-t_i)\rangle \\
=& \nu t_{\text{obs}}\int_0^{t_{\text{obs}}} \frac{\mathrm{d}t_i}{t_{\text{obs}}}\langle \ell^3\rangle_\pi \delta(t_1-t_i)\delta(t_2-t_i)\delta(t_3-t_i)
\end{aligned} \tag{14}$$

where we have used that on average the sum over i has νt_{obs} terms, and that each t_i is uniform over $[0, t_{\text{obs}}]$. Higher cumulants are of course harder to obtain in this direct manner (the third cumulant and the third moment coincide where the first moment vanishes, which is the case here because of $\langle \ell\rangle_\pi = 0$).

3. Under which conditions (on ν and on the random numbers ℓ) does η become a Gaussian white noise?

Solution: The cumulant generating function is $W[h] = \nu\int_0^{t_{\text{obs}}} \mathrm{d}t\left(\langle \mathrm{e}^{h(t)\ell}\rangle_\pi - 1\right)$ and the second cumulant is

$$\langle \eta(t)\eta(t')\rangle = \nu\langle \ell^2\rangle_\pi\delta(t-t') \tag{15}$$

Hence we need $\nu\langle \ell^2\rangle_\pi$ finite, but we want all higher cumulants to vansih, or $\nu\langle \ell^n\rangle_\pi \simeq 0$ for $n\geq 3$. If ℓ_0 is the typical scale of ℓ, then $\langle \mathrm{e}^n\rangle_\pi \sim \ell_0^n$ and we must have $\nu\ell_0^n \to$ while $\nu\ell_0^2$ fixed. This is achieved by taking $\nu\to+\infty$ and $\ell_0\to 0$ with $\nu\ell_0^2$ fixed.

4. Let $\Delta\eta = \int_t^{t+\Delta t}\eta(t')\mathrm{d}t'$. Express $\lim_{\Delta t\to 0}\frac{\langle \Delta\eta^k\rangle}{\Delta t}$ in terms of the moments of ℓ.

Solution: We have that $\Delta\eta = \sum_i \ell_i$, where the sum is over the number of points i that can be found in the $[t, t+\Delta t]$ interval.

$$\langle \Delta\eta^k\rangle = \langle \sum_{i_1,\dots,i_k}\langle \ell_{i_1}\dots\ell_{i_k}\rangle_\pi\rangle \tag{16}$$

and the external brackets bear on the number of points i in the $[t, t + \Delta t]$ interval, which is a Poisson distribution with mean $\nu \Delta t$. When all indices $i_1 = \ldots = i_k$ are equal, this is easy to evaluate:

$$\langle \sum_{i_1} \langle \ell_{i_1}^k \rangle_\pi \rangle = \nu \Delta t \langle \ell^k \rangle_\pi \tag{17}$$

but if all indices are equal, but one, we have to evaluate

$$\langle \sum_{i_1, i_2 \neq i_1} \langle \ell_{i_1}^{k-1} \rangle_\pi \langle \ell_{i_2} \rangle_\pi \rangle = (\nu \Delta t)^2 \langle \ell^{k-1} \rangle_\pi \ell \rangle_\pi \tag{18}$$

and this is of order Δt^2. In the $\sum_{i_1, \ldots, i_k}$ summation, the term that dominates the average in the $\Delta t \to 0$ limit is the one in which all indices are equal, hence $\lim_{\Delta t \to 0} \frac{\langle \Delta \eta^k \rangle}{\Delta t} = \nu \langle \ell^k \rangle_\pi$.

2.3 Translational Brownian motion

In the overdamped and Markov approximation, the Brownian motion of a particle in a bath is described by a stochastic differential equation of the form

$$\dot{x} = \sqrt{2\mu T} \eta, \; \langle \eta(t) \eta(t') \rangle = \delta(t - t') \tag{19}$$

where η is a Gaussian white noise. We work in one space dimension for simplicity.

1. What is the diffusion constant D of the particle? What would be the expression of D in higher space dimension if each component of the position independently evolved as in Eq. (19).

2. Is x a stationary process? Determine $\langle x(t) x(t') \rangle$.

3. In a variety if situations pertaining to mathematical finance [25], or the physics of disordered systems [6] or to nonequilibrium chemical processes [16], one is led to consider an observable z which is an exponential functional of a Brownian motion:

$$z(t) = \int_0^{+\infty} \mathrm{e}^{-t + g x(t)} \mathrm{d}t \tag{20}$$

where g is a given constant. Determine $\langle z \rangle$ as a function of g and D (when $\langle z \rangle$ actually exists).

Solution: We have that $\langle \mathrm{e}^{g x(t)} \rangle = \mathrm{e}^{\frac{g^2}{2} 2Dt}$ hence $\langle z \rangle = \frac{1}{1 - g^2 D}$.

4. Determine $\langle z^2 \rangle$ as a function of $g^2 D$.

Solution: Because x is a Gaussian process with zero average we have that

$$\langle e^{gx(t)+gx(t')} \rangle = e^{\frac{g^2}{2}\langle (x(t)+x(t'))^2 \rangle} \tag{21}$$

so that, using $\langle x^2(t) \rangle = 2Dt$, $x\langle x^2(t') \rangle = 2Dt'$ and $\langle x(t)x(t') \rangle = 2D \min\{t, t'\}$ we directly arrive at

$$\langle z^2 \rangle = \int \mathrm{d}t\mathrm{d}t' e^{-t-t'+Dg^2(t+t'+2\min\{t,t'\})} = \frac{1}{(1-g^2D)(1-2g^2D)} \tag{22}$$

which is well-defined for $g^2 D < 1/2$. Finding the whole pdf of z is a little more involved.

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2.4 Langevin equation for a classical magnet

In the book by Coffey and Kalmykov [4] a large section is devoted to the study of rotational Brownian motion. This applies to the modeling of mesoscopic magnets, electric dipoles, but also to liquid crystals or other elongated particles such as an E. coli bacterium. The physics of magnetism is of course of quantum origin, but it is possible to describe the thermal fluctuations of a small single domain of ferromagnetic particles using classical physics. Because information storage relies on the magnetic domains being able to retain their magnetization, in order to increase the amount of information stored, achieving ever smaller domains is of course a desired goal. However, for small enough domains, the possibility that thermal fluctuations actually destroy existing order cannot be overlooked [3, 5]. Hence the importance of understanding the interplay of thermal fluctuations with the dynamics of a single mesoscopic magnet.

In this problem, we first explore how rotational Brownian motion is described by a Langevin equation, and then we look at the specifics of the dynamics of a mesoscopic magnet.

1. Consider a particle characterized by its position $\mathbf{R}$ and by a unit vector $\mathbf{u}$ that fully characterizes its orientation in space. The particle is assumed to be in contact with a thermal bath, and it is in some external potential $V_0(\mathbf{R}, \mathbf{u})$ that may act both on the translational and rotational degrees of freedom. Under what assumptions can we write the effective dynamics of $\mathbf{R}$ as stochastic differential equation of the form

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = -\mu \frac{\partial V_0}{\partial \mathbf{R}} + \sqrt{2\mu T}\boldsymbol{\eta} \tag{23}$$

where $\boldsymbol{\eta}$ is a Gaussian white noise with independent components, $\langle \eta^\mu(t)\eta^\nu(t') \rangle = \delta(t-t')$.

Solution: We identify an overdamped Langevin equation without memory term, which means that not only we are working at low Reynolds numbers (high viscosity, inertia is negligible) but also that time-scales related to the bath are much shorter than those characterizing the system of interest.

2. Let I be the inertia tensor of the particle and $\boldsymbol{\Omega}$ its rotation vector. Then the angular momentum $\mathbf{L} = I\boldsymbol{\Omega}$ evolves according to

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_b \tag{24}$$

where $\boldsymbol{\Gamma}_0$ and $\boldsymbol{\Gamma}_b$ are the torques exerted by the external operator and by the bath particles, respectively. Connect $\boldsymbol{\Gamma}_0$ to V_0.

Solution: This is basic mechanics: $\boldsymbol{\Gamma}_0 = -\mathbf{u} \times \partial_{\mathbf{u}} V_0$.

3. When interested in the statistics of $\mathbf{R}$ and $\mathbf{u}$, it is possible to replace the individual degrees of freedom of the bath particles by the combination of a deterministic contribution and a random one. In your opinion, how does the average torque created by the bath, $\langle \boldsymbol{\Gamma}_b \rangle_b$, connect to the angular degrees of freedom? You should work within the Markov approximation where the time scale of the bath degrees of freedom is much shorter than any relevant time scale related to the particle of interest.

Solution: By analogy to the translational case, we expect there will be a viscous damping term of the form $\langle \boldsymbol{\Gamma}_b \rangle_b = -\zeta \boldsymbol{\Omega}$.

4. Let $\boldsymbol{\xi} = \boldsymbol{\Gamma}_b - \langle \boldsymbol{\Gamma}_b \rangle_b$. What can you say about the statistics of $\boldsymbol{\xi}$? How about $\langle \xi^\mu(t) \xi^\nu(t') \rangle$?

Solution: We expect Gaussian statistics and white correlations.

5. Express $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ in terms of $\boldsymbol{\Omega}$ and $\mathbf{u}$.

Solution: By definition of $\boldsymbol{\Omega}$ we have $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \mathbf{u}$.

6. Find a Langevin equation for $\mathbf{u}$ within the approximation in which the time scale τ_{inertia} related to inertia is also much shorter than any relevant time scale related to the particle of interest. Write the stochastic differential equation for $\mathbf{u}$ in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \text{deterministic term } + \underbrace{\boldsymbol{\lambda}}_{\text{noise term with zero mean}} \tag{25}$$

What is the amplitude of the correlations of $\boldsymbol{\lambda}$ in terms of $\mathbf{u}$?

Solution: The inertial time scale is controled by $\tau_{\text{inertia}} = I/\zeta$. If it is short, then we can approximation the equation of motion by

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_b \simeq \mathbf{0} \tag{26}$$

with $\boldsymbol{\Gamma}_b = -\zeta\boldsymbol{\Omega} + \boldsymbol{\xi}$. We can they write that

$$\zeta\boldsymbol{\Omega} = \boldsymbol{\Gamma}_0 + \boldsymbol{\xi} \tag{27}$$

and taking the cross product with $\mathbf{u}$ to the right leads to

$$\zeta\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \boldsymbol{\Gamma}_0 \times \mathbf{u} + \boldsymbol{\lambda} \tag{28}$$

with $\boldsymbol{\lambda} = \boldsymbol{\xi} \times \mathbf{u}$ which has correlations

$$\langle \lambda^\mu(t)\lambda^\nu(t)\rangle = C(\mathbf{u}^2\delta^{\mu\nu} - u^\mu u^\nu)\delta(t-t') \tag{29}$$

where C is the amplitude of $\boldsymbol{\xi}$.

7. Assuming one can manipulate $\mathbf{u}$ as if it were a smoothly differentiable function, does your equation respect the $\mathbf{u}^2 = 1$ constraint?

Solution:

8. We now get back to the magnetic problem of interest and consider a magnet with dipole $\mathbf{m} = m_0\mathbf{u}$ in some external field $\mathbf{B}$ (m_0 is fixed). The energy is $V_0 = -\mathbf{m}\cdot\mathbf{B}$. Translational degrees of freedom are not included in the description. Express $\boldsymbol{\Gamma}_0$ in terms of $\mathbf{m}$ and $\mathbf{B}$.

Solution: We have $\boldsymbol{\Gamma}_0 = \mathbf{m} \times \mathbf{B}$.

3 Stochastic Calculus

3.1 Differential calculus likes Stratonovich discretization

Consider a Langevin equation $\frac{\mathrm{d}x}{\mathrm{d}t} = A + B\xi$, where ξ is a Gaussian white noise with correlations $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$. The multiplicative noise B is understood with the α discretization rule, namely according to

$$\Delta x = x(t+\Delta t) - x(t) = A(x(t)+\alpha\Delta x)\Delta t + B(x(t)+\alpha\Delta x)\Delta\xi,\ \Delta\xi = \int_t^{t+\Delta t}\mathrm{d}t'\xi(t') \tag{30}$$

In the above equation, in the rhs, the first term is of order Δt while the second one is of order $\sqrt{\Delta t}$.