Chapter 5

Thermal Ratchets and Stochastic Motors

5.1 Motivations

5.1.1 Back to the XIXth century, and into the XXIst

Let's examine a macroscopic system, characterized by its usual state variables (pressure P, volume V, temperature T) that is subjected to a sequence of transformations. Start from state A with temperature T_h, and let the system spontaneously expand to reach state B. By doing so it will cool down, so we decide to inject some heat into the system to maintain its temperature T_h while it expands from V_A to V_B. By expanding, we retrieved some work that the system actually gives away to the outside world. Then, we isolate the system and let the expansion continue. Now the system cools down to temperature T_c during this adiabatic step. This is an entropic expansion, up to state C at V_C. Then we compress the system to state D by keeping its temperature constant at T_c, which means the system must release heat to the outside world. Finally we get back to the initial state A via an adiabatic isoentropic compression. The first principle tells us that $\Delta U_{A\to A} = Q_{A\to B} + Q_{C\to A} + W = 0$ and the second principle tells us that $\Delta S_{A\to A} = \frac{Q_{A\to B}}{T_h} + \frac{Q_{C\to D}}{T_c}$, hence the efficiency $\eta = \frac{\text{what we get}}{\text{what we spend}}$ is given by $\eta = -\frac{W}{Q_{A\to B}}$, so that

$$\eta = 1 - \frac{T_c}{T_h} \tag{5.1}$$

and $\eta \to 0$ as $T_c \to T_h$. For macroscopic systems, there is simply no way to extract work from a single thermal bath.

Let us now enter the XXIst century and see if our conclusion can change. We follow [86] for the experiment, and [104] for the theory. Let's quote the authors of the experimental work:

> The Carnot cycle consists of two isothermal processes, where the working substance is respectively in contact with thermal baths at different temperatures T_h and T_c, connected by two adiabatic processes, where the substance is isolated and heat is not delivered nor absorbed. An external parameter is changed in such a way that

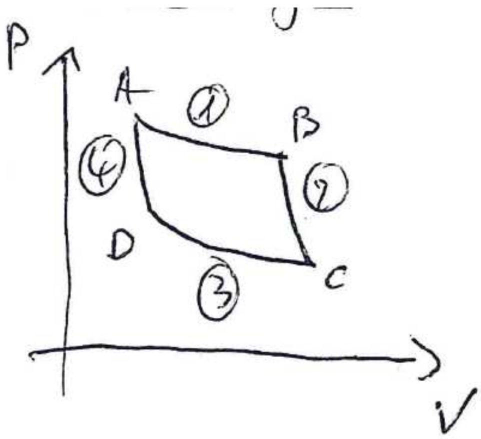

Figure 5.1: The cycle is run clockwise, so that $\oint P\mathrm{d}V > 0$ and thus $W < 0$, in accordance with the idea that some useful work is actually provided by the system. Sketch courtesy of J. Tailleur.

> the whole cycle is carried out reversibly. Following this scheme, one could devise a progressing miniaturization of a Carnot engine and eventually reproduce the cycle with a single Brownian particle. In fact, a variety of thermodynamic processes and even a complete Stirling cycle have been already implemented in the mesoscale using micro-manipulation techniques. Interestingly, the exchange of energy between the particle and its surrounding environment becomes stochastic at the microscale and yet one can rigorously define work, heat and efficiency, within the framework of the recently developed stochastic thermodynamics.
>
> The experimental realization of a Carnot cycle with a single Brownian particle has remained elusive owing to the difficulties of implementing an adiabatic process. In particular, it is not clear how to isolate a particle from the surrounding fluid. A more feasible strategy is to simultaneously change the temperature and the external parameter keeping constant the Shannon entropy of the particle. However, the necessary fine-tuning of the temperature is an experimental challenge as well. Here we construct a Brownian Carnot engine putting forward an experimental technique that allows precise control of both the effective temperature and the accessible volume of a single microscopic particle. We use a particle with an inherent electric charge and apply a noisy electrostatic force that mimics a thermal bath. In this way, we can achieve temperatures ranging from room temperature (no electrostatic force) up to hundreds or even thousands of kelvins, far above the boiling point of water.
>
> The working substance of our engine is a single optically trapped colloidal particle immersed in water.

For small displacements of the particle within the optical trap, the latter can be modeled by an external potential $V(x,t) = \frac{k(t)}{2}x^2$, where the stiffness k is controled in much the same way as the volume was controled in the original Carnot cycle. The states $3, 4, 1, 2$ match the

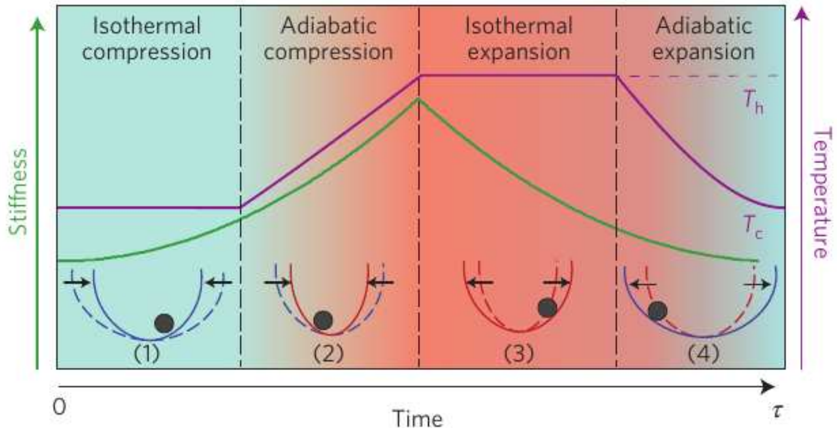

Figure 5.2: From [86].

A, B, C, D of the Carnot cycle. Let's investigate the energy balance $\mathrm{d}V = \delta W + \delta Q$ where $\delta W = \frac{1}{2}x^2\mathrm{d}k$ and $\delta Q = T\mathrm{d}s$, with $S(t) = -\ln p_{\mathrm{eq}}(x(t))$. First, let's make sure that this is true!

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \dot{k}\frac{x^2}{2} + kx\dot{x} \tag{5.2}$$

and we see that if $p_{\mathrm{eq}}(x) = \mathrm{e}^{-\beta kx^2/2}/Z$ then $s(t) = -\beta\frac{kx(t)^2}{2} + \ln Z$ so that $\mathrm{d}s = -\beta kx\dot{x}\mathrm{d}t$. This is consistent with the idea that $0 = -kx - \dot{x} + \sqrt{2T}\eta$ and thus the work of the external force $-kx\dot{x}\mathrm{d}t$ is also the work done by the particle on the bath, namely $-kx\dot{x}\mathrm{d}t = -(-\dot{x} + \sqrt{2T}\eta)\dot{x}\mathrm{d}t = \delta Q$. From here on it should be pretty clear that taking averages will not modify anything to the Carnot efficiency!

5.1.2 Feynman's ratchet and pawl's paradox

In chapter 46 of the first volume of the Feynman Lectures on Physics [38], Feynman tricks the reader with the following thought experiment in Fig. **??**.

The question is whether the wheel will rotate anticlockwise. If the wheel had a net rotation, say with angular velocity ω_0, then this would mean that it could raise up a small body of mass m, by reducing its velocity to ω_m. We would then have an equilibrium system performing work, which contradicts the second principle of thermodynamics which states that work can be extracted only with two sources (and the highest efficiency is obtained with the Carnot cycle). We will later explore what occurs when $T_1 \neq T_2$ in the right compartment. At first sight, it appears that a fluctuation could very well lead to the wheel hopping ahead and thus peforming a net motion. As noted by Feynman, the first issue comes up when describing the pawl. For it to return in place and prevent any backward motion of the wheel, there has to be some spring pulling it back. Suppose also that the parts of the device are perfectly elastic, then this would mean that upon returning back to its initial position, the pawl would actually keep

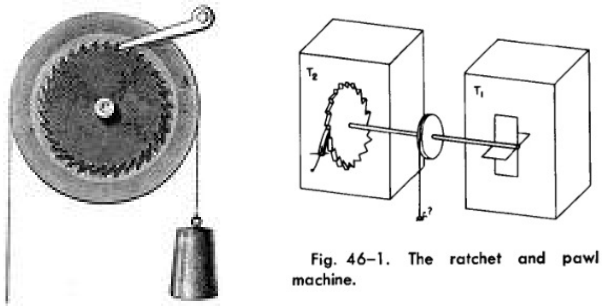

Figure 5.3: Left: A freely rotating ratchet (toothed wheel) is immersed in an equilibrium gas at temperature T, and a pawl is used to block the backward motion of the wheel. The net motion is used to extract work to raise a body of mass m. Right: The original discussion by Feynman, based on the same principl;e, and with $T_1 = T_2$ to begin with.

bouncing. Then, this leaves room for a fluctuation to move the wheel backward as the pawl is temporarily lifted. One understands that for the pawl not to bounce, one needs some sort of dissipation. The dissipated energy will simply heat the wheel up, which will become hotter and hotter. Some of the heat will be absorbed by the surrounding gas, but not forever. As things get hotter, such a rare even by which a thermal fluctuation raises the pawl and moves the wheel backward is more and more likely. In [98] the authors have criticized some aspects of Feynman's original discussion.

We return to the right device in figure 5.3 where we now allow the two temperatures to be unequal with $T_2 < T_1$. We proceed with a standard macroscopic thermodynamics analysis. Because the wheel is cold and the fluctuations of the pawl are rather infrequent, it will be hard for the pawl to attain energy ε. In contrast, since T_1 is larger, the vane will more frequently than the ratchet reach the energy ε. The whole device will then indeed proceed with a net motion, as expected. But can it lift weights? A small mass m is hung on the ratchet-vane axis, thus exerting a torque Γ. Our questions are: how much weight can it lift and how fast does it go? In the following, ε refers to the energy necessary to lift the pawl.

We consider the forward motion. The energy needed for rotating the wheel by an angle θ is $\varepsilon + \Gamma\theta$, and this energy is taken at a rate $\gamma e^{-\beta_1(\varepsilon+\Gamma\theta)}$, where γ is setting the time-scale. We now look at the opposite event. The rate at which the backward motion of the wheel occurs is $\gamma e^{-\beta_2\varepsilon}$. The work W released is then $W = \Gamma\theta$. Work is indeed released since the wheel slips backward. During the latter step, the energy given to the vane at T_1 is $\varepsilon + \Gamma\theta$. In the foward motion, the needed energy is $\varepsilon + \Gamma\theta$ and it is fully taken from the vane. Work performed accounts for $\Gamma\theta$ and ε is given to the ratchet. In the backward motion, up to a sign reversal, the same energy ε is taken from the ratchet, the work $\Gamma\theta$ is released and $\Gamma\theta + \varepsilon$ is given to the vane. If the pawl is lifted up accidentally by a fluctuation, then when it falls back the spring pushes it down against the tooth, and there is a force trying to turn the wheel because the tooth is pulling on an inclined plane. This force is doing work and so is the force due to the weights. Both together make the total force and all the energy which is slowly released appers in the form of heat at the vane. For a particular value of the torque (or of the mass) the rates

Figure 5.4: A wheat spike.

will be equal. If an infinitesimal weight is added to the string, work is done on the machine, and if it is removed, heat is taken from the vane and put into the wheel. The condition at which there is perfect balance is $\frac{\varepsilon+\Gamma\theta}{T_1} = \frac{\varepsilon}{T_2}$, and it corresponds to a Carnot reversible cycle. If the machine is slowly lifting the weight, $Q_1 = \varepsilon + \Gamma\theta$ is taken from the vane and $Q_2 = \varepsilon$ is delivered to the wheel: $Q_1/Q_2 = T_1/T_2$. The work-to-energy-taken-from-the-vane ratio is $\frac{W}{Q_1} = \frac{\Gamma\theta}{\varepsilon+\Gamma\theta}$. No more work than this limiting ratio obtained from reversible conditions can be extracted. If one had $T_1 = T_2$, the angular velocity ω is θ times the rate at which a jump occurs. The forward motion has a rate $\gamma e^{-\beta_1(\Gamma\theta+\varepsilon)}$ and the backward one $\gamma e^{-\beta_1\varepsilon}$, hence

$$\omega = \gamma\theta e^{-\beta_1\varepsilon}(e^{-\beta_1\Gamma\theta} - 1) \tag{5.3}$$

which leads to $\omega(\Gamma)$ being a decreasing function of Γ, with $\omega(\Gamma < 0) > 0$ and $\omega(\Gamma > 0) < 0$. If $\Gamma > 0$, which corresponds to driving the wheel backward, the backward velocity approaches a constant $-\theta\gamma e^{-\beta_1\varepsilon}$, while $\omega(\Gamma \to -\infty) \to +\infty$. If now we take the weight away and reinstate the two unequal temperatures, then, if $T_2 < T_1$ we have forward motion, as expected. The reverse case $T_2 > T_1$ leads to a backward motion, which may be harder to conceive. A hot ratchet and pawl is ideally built to go around in a direction exactly opposite to that for which it was originally designed. Indeed, with lots of heat, the ratchet keeps bouncing. If the pawl, for a moment, is on the incline somewhere then it pushed the inclined plane sideways. But is is always pushing an inclined plane, because if it happens to lift up high enough to get past the point of a tooth, then the inclined plane slides by and the pawl comes down again on an inclined plane. If the two temepratures are equal, there is no more propensity to turn one way or another.

The wheat spike

Consider a wheat spike as shown in figure 5.4. If placed in between two layers of clothing, say in between the sleeve of a coat and that of the underlying sweater, the spike will rise up the arm due to the random motion of the arm during the walk. There is a net displacement in a preferred direction, but the spike is not inequilibrium, because of an external force pulling it upwards against gravity. The force acting on the spike is random, in that it fluctuates, but the latter fluctuations are rectified, since the spike benefits only from the component of the force that drives it up. Whence the question we would like to answer: under what conditions can a fluctuating force be led to perform a net work by rectification. Curie's Principle states that the symmetry of a cause is always preserved in its effects. This makes us believe that the wheat spike being asymmetric, its displacement along a preferred direction was expected. If that is so, why not repeating the argument for the ratchet and the pawl? The answer is no because the reversibility of the microscopic evolution equations –a symmetry property– will be preserved, which is in contradiction with the wheel rotating in a favored direction. However, out of equilibrium, in a stationary state with irreversible dynamics, the rectification of fluctuations becomes a possibility.

In what follows we'll explore several routes. One is inspired by the Carnot experiment (or the ratchet and pawl) and we will drive an energy flux through the system, say by imposing contact with thermal baths at different temperatures. Another one consists in breaking time-reversal at the microscopic level.

5.2 Fluctuating forces

5.2.1 No current in equilibrium

Our presentation is based on [80]. Before anything, let's examine what is happening with a particle in equilibrium evolving in a sawtooth potential as in Fig. 5.5. The equation of motion

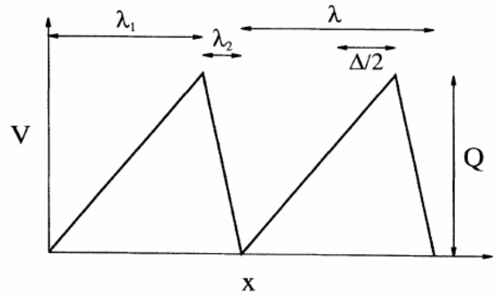

FIG. 1. A plot of the piecewise linear potential $V(x)$ as a function of position x. The width of each segment is called λ_1 and λ_2. The period of the potential is $\lambda=\lambda_1+\lambda_2$ and the symmetry breaking amplitude is $\Delta=\lambda_1-\lambda_2$.

Figure 5.5: Taken from [80].

is $\dot{x} = -V' + \sqrt{2T}\eta$. Is there any particle current in the steady-state? If there would be one, $j = -V'p - T\partial_x p$ would be nonzero, and thus

$$p(x) = \mathrm{e}^{-\beta V(x)}\left[p(0)\mathrm{e}\beta V(0) - \beta j\int_0^x \mathrm{d}y\mathrm{e}^{\beta V(y)}\right] \tag{5.4}$$

with $\int \mathrm{d}xp(x) = 1$. The periodicity condition $p(0) = p(\ell)$, and the fact that $V(0) = V(\ell)$, lead to $j = 0$. In the thermal equilibrium problem, the potential's asymmetric shape alone is of no help in producing a current.

5.2.2 Brownian motion in an asymmetric potential, out of equilibrium (flashing ratchet)

One way to drive the same system out of equilibrium is to replace the constant thermal bath by an oscillating temperature $T(t) = T_0 + T_1\cos^2(\pi t/\tau)$. It remains to understand how such anoscillating thermal contact may lead to a nonvanishing current. Since an exact analytical solution is not available, we shall confine our analysis to an even simpler situation in which

$$T(t) = \begin{cases} 0 & \text{for } 0 < t < \tau/2 \\ T_0 & \text{for } \tau/2 < t/\tau \end{cases} \tag{5.5}$$

which is τ-periodic. We shall further assume that $k_{\mathrm{B}}T_0 \gg V(x)$. The particle starts at $t = 0$ from $x = x_0$. After a half-period, at $t = \tau/2$, at temperature $T = 0$, the particle lies at the potential well closest to x_0, which according to figure 5.6, is at $x = x_1$. At $t = \tau$, for the

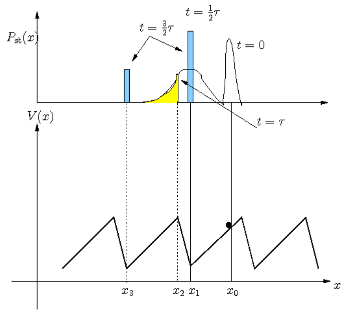

Figure 5.6: Periodic potential.

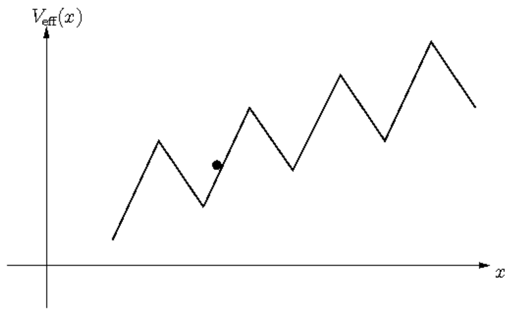

Figure 5.7: Periodic potential with a negative and constant force f.

last half-period, the particle has been executing a random walk with a temperature $k_B T \gg V$, which leads to a wide Gaussian pdf centered around x_1. At $t = \frac{3}{2}\tau$, the particle again moves deterministically towards the closest minimum, because it has a larger probability overlap with the descending slope of the potential towards x_3. Actually, if at $t = \tau$ the particle is left of x_2, then it willmove towards x_2, and to x_1 is the particle is right of x_2. Between $t = \frac{1}{2}\tau$ and $t = \frac{3}{2}\tau$, one can see that a net probability transport has occurred to the left. Such a phenomenon survives if the temperature actually follows the periodic laws mentioned above, or if it varies randomly around a given threshold.

The same system is now in addition subjected to a constant linear force, thereby feeling the effctive potential $V_{\text{eff}}(x) = V(x) - fx$, with a negative load force $f < 0$ in figure 5.7. If the temperature is constant, $T(t) = T_0$ then we just return to the solution of the Fokker-Planck equation (**??**) in which we replace $V(x)$ with $V_{\text{eff}}(x)$. A nonzero current $J = \text{cst} \times f$ is now present, as shown in figure 5.8. We found in the previous section that in the absence of a load force, there already existed a nonzero current J_0, as indicated in figure 5.8. With a varying temperature, if the load force f is now imposed, we expected the $J(f)$ curve to appear roughly like the dashed line in figure 5.8 (what is shown is actually the time-averaged current), which means that there exists a regime of the load force in which the current is actually opposed to the force. The system is able to provide work against the external force: this is a molecular motor.

Note that the stationary solution in the presence of a constant temperature thermal bath

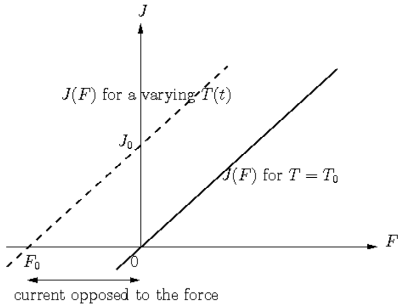

Figure 5.8: The constant temperature solution is shown is full line. The dashed line stands for the current in the presence of a varying temperature.

is given by

$$P_{\rm st}(x) = Z^{-1}\int_0^1 \mathrm{d}y \mathrm{e}^{\beta W(y,x)} \tag{5.6}$$

where

$$W(y,x) = V(y) - V(x) + \begin{cases} \int_y^x \mathrm{d}z f(z) & \text{if } y \leq z \\ \int_y^1 \mathrm{d}z f(z) + \int_0^x \mathrm{d}z f(z) & \text{if } y > x \end{cases} \tag{5.7}$$

is the work performed by the applied forces along the positively oriented path. The normalization factor reads $Z = \int_0^1 \int_0^1 \mathrm{d}x \mathrm{d}y \mathrm{e}^{\beta W(y,x)}$ and the net probability current is easily obtained by dividing $j = -\partial_x P_{\rm st} + \beta(f - V')P_{\rm st}$ by $P_{\rm st}$ and then by integrating the resulting equation over $[0, 1]$, which gives

$$j = \beta \frac{W_{\rm cyle}}{\int_0^1 \mathrm{d}x/P_{\rm st}(x)} \tag{5.8}$$

where $W_{\rm cycle} = \int_0^1 \mathrm{d}z f(z)$ is the work over a complete cycle. For a constant force, this directly leads to $j = \beta f / \int_0^1 \mathrm{d}x/P_{\rm st}(x)$. In the experiment carried out by Mehl *et al.* [88] the potential is $V(x) = \frac{1}{2}V_0 \sin(x/R)$ (with $V_0 = 58k_{\rm B}T$ and $f = 30k_{\rm B}T$ is constant). The related velocity is 7 μm/s. The system is a colloidal particle of radius 0.65 μm, immersed in water, and trapped within a three-dimensional torus of radius R. The angular coordinate x of the particle is tracked between $-\pi R$ and $+\pi R$, with a time and spatial accuracy of 10 nmm and 15 ms respectively. The force f is exerted by optical tweezers.

5.2.3 Gallavotti-Cohen relation in the flashing ratchet

In this section, we draw freely from Lacoste and Mallick [70]. We take the motor to be a small particle moving in a one-dimensional space. At the initial time $t = 0$, the motor is trapped within one the wells of periodic asymmetric potential with spatial period a. Between 0 and τ_f, the asymmetric potential is erased and the particle diffuses freely and isotropically in contact with a thermostat at temperature T. At the switching time τ_f, the asymmetric potential is re-impressed and the motor slides down the nearest potential well in which it gets trapped, due to dissipation. The motor has the largest probability to fall within the same well it was in before the potential was erased, but there is a nonzero probability that it ends up to the

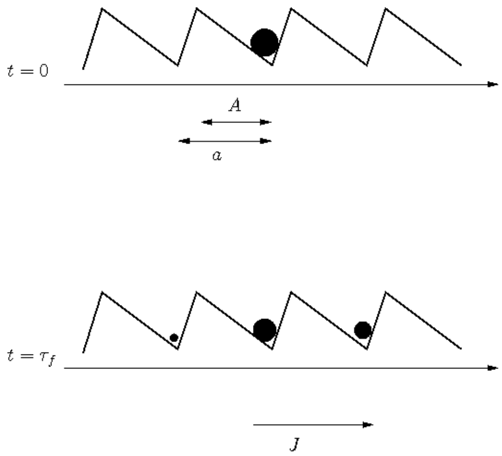

Figure 5.9: The flashing ratchet model.

right hand side well, and yet an even smaller probability to fall with the left-hand side one, as shown in figure 5.9. The fact that $a - A < A$ expresses that the potential is asymmetric. In the lower part of figure 5.9, the sizes of the disks are proportional to the presence probabilities of the particle. In practce, in order to move, ATP fuel is eaten up, at a rate r, which is broken into ADP and P,

$$\text{ATP} \leftrightarrow \text{ADP} + \text{P} \tag{5.9}$$

The chemical energy released through the ATP hydroluysis allows the motor to detach itself from the filament it was bound to. This detachment process corresponds to the basic erasing mechanism, while the reattachment is equivalent to re-impressing the potential after τ_f. Hence the motor undergoes chemistry driven changes between strongly and weakly bound states. Energy barriers are overcome by the coupling betwwen chemistry and the interaction with the filament. The filament is a polar object, which is reflected in the asymmetric interaction potential. In general, the motor is subjected to an external force f_{ext} which further tilts the potential. If ATP is in excess, $\Delta\mu = \mu_{\text{ATP}} - \mu_{\text{ADP}} - \mu_{\text{P}} > 0$, and we shall denote $E = \beta\Delta\mu$, with

$$E = \ln \frac{[\text{ATP}]}{[\text{ADP}][\text{P}]} \frac{[\text{ADP}]_{\text{eq}}[\text{P}]_{\text{eq}}}{[\text{ATP}]_{\text{eq}}} \tag{5.10}$$

The basic question is to determine the velocity of the motor $v(f_{\text{ext}}, E)$ and the ATP consumption rate $r(f_{\text{ext}}, E)$ as a function of the applied forces.

In the flashing ratchet model, the switching of one potential to the other is sudden and occurs at random times generated by a Poisson process. The state of the motor is encoded in its position x and by two internal states $i = 1$ and 2 corresponding to distinct internal states. One of the states –the high energy one– corresponds to the motor being bound to the filament, and the other one –the low energy one– to a configuration where both heads are bound. In the previous discussion, U_1 is a sawtooth potential and U_2 is zero. We extend somexhat the discussion by allowing U_1 and U_2 to be arbitray asymmetric potentials with spatial period a.

The motor dynamics is given by the following Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\gamma F - \gamma \sum_i U_i \delta_{\zeta(t),i} + \sqrt{D}\xi \tag{5.11}$$

where $D = \sqrt{\gamma k_{\mathrm{B}} T}$, ζ is a dichotomous noise that can exist in two states 1 and 2, and xi is a Gaussian white noise of unit variance. The switching rates of ζ are position dependent and are given by $\omega_1(x)$ for the $1 \to 2$ transition, and by $\omega_2(x)$ for the $2 \to 1$ transition. The external force acting on the motor is denoted by F. The probability to find the motor in state i evolves according to

$$\partial_t P_1 + \partial_x J_1 = -\omega_1 P_1 + \omega_2 P_2,\ \partial_t P_2 + \partial_x J_2 = -\omega_2 P_2 + \omega_1 P_1 \tag{5.12}$$

where $J_i = -D(\partial_x P_i + k_{\mathrm{B}} T(\partial_x U_i - F)P_i)$. The transition rates are given by standard kinetics for chemical reactions,

$$\omega_1(x) = \left(\omega(x) + \psi(x)\mathrm{e}^{E}\right)\mathrm{e}^{\beta(U_1 - f_{\mathrm{ext}} x)},\ \omega_1(x) = \left(\omega(x) + \psi(x)\right)\mathrm{e}^{\beta(U_2 - f_{\mathrm{ext}} x)} \tag{5.13}$$

with $f_{\mathrm{ext}} = Fa$ and where ω describes standard thermal transitions, while the extra ψ-dependent piece express the bias imposed by ATP hydrolysis. These two functions are periodic, but are otherwise unspecified. In the absence of any hydrolysis, there is detailed balance, $\omega_2/\omega_1 = \mathrm{e}^{-\beta U_1}/\mathrm{e}^{-\beta U_2}$. In the presence of hydrolysis, there is a generalized detailed balance condition that imposes a sort of local detailed balance,

$$\frac{\omega_2}{\omega_1} = \mathrm{e}^{-\beta(U_1 - U_2) - E} \tag{5.14}$$

For $F = 0$ and $E = 0$, the system is in equilibrium and there is no net displacement of the motor.

Let $P_i(x, q, t)$ be the probability that at time t the motor is in internal state i at position x and that q chemical units of ATP have been consumed. The master equation for $\hat{P}_i(u, \lambda, k, t) = \sum_q \mathrm{e}^{-\lambda q - k(n+u)} P_i((n+u)a, q, t)$ has the linear evolution operator

$$\mathbb{W}(\lambda, k, x) = \begin{pmatrix} \mathbb{L}_1 - \omega_1 & 0 \\ 0 & \mathbb{L}_2 - \omega_2 \end{pmatrix} + \begin{pmatrix} 0 & \omega \mathrm{e}^{\beta(U_2 - fx)} + \psi \mathrm{e}^{\beta(U_2 - fx)}\mathrm{e}^{\lambda} \\ \omega \mathrm{e}^{\beta(U_1 - fx)} + \psi \mathrm{e}^{\beta(U_1 - fx) + E}\mathrm{e}^{-\lambda} & 0 \end{pmatrix} \tag{5.15}$$

where

$$\mathbb{L}_i(q)\bullet = D_0 \partial_u^2 \bullet + \partial_u(\beta(U_i - fu)\bullet) + 2q\partial_u \bullet + q\beta(U_i' - f)\bullet + q^2 \bullet \tag{5.16}$$

The diagonal matrix $Q = \mathrm{diag}(\mathrm{e}^{-\beta U_1}, \mathrm{e}^{-\beta U_2})$ verifies

$$Q^{-1}\mathbb{W}(\lambda, q)Q = \mathbb{W}^{\dagger}(E - \lambda, f - q) \tag{5.17}$$

which leads to the Gallavotti-Cohen symmetry for the largest eigenvalue of $\mathbb{W}$.

5.2.4 With a fluctuating force

We now consider a periodic sawtooth potential with period λ as shown in Fig. 5.5 which we subject, in addition to a force $F(t)$ that has zero ensemble average (if it is fluctuating) or zero time-average (if it is deterministic). It is useful to begin with a constant F only to get back to a time-varying F afterwards, which is fine as long as F varies slowly enough.

When F is a constant, the Fokker-Planck equation $\partial_t p = \partial_x((V' - F)p) + T\partial_x^2 p$, and the current $j = -T\partial_x p + Fp - V'p$ allows us to find that

$$p(x) = p(0)\mathrm{e}^{\beta(Fx-V(x))} - \beta j \int_0^x \mathrm{d}y \mathrm{e}^{-\beta(Fx-V'(x))}\mathrm{e}^{\beta(V(y)-Fy)} \tag{5.18}$$

On the $[0, \lambda_1]$ segment we have $V(0) = 0$, $V(\lambda_1) = Q$ and thus $V(x) = \frac{Qx}{\lambda_1}$ while on the $[\lambda_1, \lambda]$ segment we have $V(x) = \frac{\lambda-x}{\lambda_2}Q$. We have so far two unknowns, $p(0$ and j. One is fixed using that $p(0) = p(\lambda)$ and the other one by imposing $\int_0^\lambda p(x) = 1$. Let's exploit periodicity,

$$p(0) = p(0)\mathrm{e}^{\beta F\lambda} - \beta j \mathrm{e}^{\beta F\lambda} \int_0^\lambda \mathrm{d}y \mathrm{e}^{\beta(V(y)-Fy)} \tag{5.19}$$

and we plit the y integral into two pieces,

$$\begin{aligned}\int_0^\lambda \mathrm{d}y \mathrm{e}^{\beta(V(y)-Fy)} &= \int_0^{\lambda_1} \mathrm{d}y \mathrm{e}^{\beta(Q/\lambda_1-F)y} + \int_{\lambda_1}^\lambda \mathrm{d}y \mathrm{e}^{\beta(\lambda Q/\lambda_2-\beta(F++Q/\lambda_2)y)} \\ &= \frac{\mathrm{e}^{\beta(Q/\lambda_1-F)\lambda_1} - 1}{\beta(Q/\lambda_1 - F)} + \mathrm{e}^{\beta\lambda Q/\lambda_2} \frac{\mathrm{e}^{-\beta(F+Q/\lambda_2)\lambda_1)} - \mathrm{e}^{-\beta(F+Q/\lambda_2)\lambda}}{\beta(F + Q/\lambda_2)}\end{aligned} \tag{5.20}$$

Simplifying further the condition $p(0) = p(\lambda)$ then leads to

$$p(0)(1 - \mathrm{e}^{\beta F\lambda}) = -j \left[\lambda_1 \frac{\mathrm{e}^{\beta(Q+\lambda_2 F)} - \mathrm{e}^{\beta\lambda F}}{Q - \lambda_1 F} - \lambda_2 \frac{1 - \mathrm{e}^{\beta\lambda_2 F+\lambda Q}}{Q + \lambda_2 F}\right] \tag{5.21}$$

and thus the connection between j and $p(0)$ is given by

$$j(F) = p(0) \frac{2\sinh(\beta\lambda F/2)}{\frac{\lambda_1}{Q-\lambda_1 F}\left(\mathrm{e}^{\beta(Q-\Delta F/2)} - \mathrm{e}^{\beta\lambda F/2}\right) + \frac{\lambda_2}{Q+\lambda_2 F}\left(\mathrm{e}^{\beta(Q-\Delta F/2)} - \mathrm{e}^{-\beta\lambda F/2}\right)} \tag{5.22}$$

where $\Delta = \lambda_1 - \lambda_2$.

At this stage we can get back to our original question by allowing from slow variations of F (think of $F(t)$ being a square periodic signal between $-F$ and $+F$), and by slow we mean that the typical period T of F should be much larger than the relaxation time τ of the process. If T is large enough, the steady state we have just worked out has time to be established. When the particle feels a change from F to $-F$, it will take at typical time τ for it to reach it new