or

$$\langle (x(t)-x(t'))^2\rangle = 2D(t-t'),\ t'\leq t \tag{3.20}$$

and thus $\Delta x = x(t+\Delta t)-x(t)$ has variance $\langle \Delta x^2\rangle = 2D\Delta t$. This immediately tells us that as $\Delta t\to 0$ the displacement Δx is typically of order $\sqrt{\Delta t}$ and thus that $\frac{\Delta x}{\Delta t}$ is of order $\Delta t^{-1/2}$, which diverges as $\Delta t \to 0$. This is the mathematical translation of the statement that a Brownian motion trajectory, in spite of being continuous everywhere, is nowhere differentiable. There is no such thing as a free lunch. If we insist on writing something like $\dot{x}$ it cannot, strictly speaking, have the usual meaning that

$$\dot{x} = \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} \tag{3.21}$$

because we know this is infinite.

In order to realize the sort of trouble that lies ahead of us, let's consider $y = x^2/2$ and pretend we can manipulate $\dot{x} = \sqrt{2D}\eta$ as we if these were smooth enough functions. Then we multiply $\dot{x} = \sqrt{2D}\eta$ by x and we get

$$\dot{y} = \sqrt{2D}\sqrt{2y}\eta \tag{3.22}$$

and though we know that $\langle y(t)\rangle = Dt$ (for $x(0) = 0$), which means that $\langle \dot{y}\rangle = D$ we hardly see how we can ever average Eq. (3.22) to actually see that result directly. In fact we shouldn't be alarmed. Only a very naive person would manipulate non-differentiable functions as it they were. And we are not naive. Let's see how to sort all this out.

3.2 Discretizing a Langevin equation

3.2.1 Procedure

Let x_0 be fixed and define x_n via the following recursion relation,

$$x_{m+1}-x_m = f(x_m)\Delta t + g(x_m)\sqrt{\Delta t}\xi_m,\ m = 0,2,\dots \tag{3.23}$$

where the ξ_m's are independent zero-mean Gaussian variables with unit variance,

$$\langle \xi_m\xi_{m'}\rangle = \delta_{mm'} \tag{3.24}$$

For the moment, the functions f and g are arbitrary and $\Delta t > 0$ will eventually be taken as small as possible. This way, one constructs a sequence $x_1, x_2\dots$ of random variables. It is tempting to define

$$x(t) = x_{t/\Delta t} \tag{3.25}$$

and to write that Eq. (3.23) amounts to

$$\Delta x = x(t+\Delta t)-x(t) = f(x(t))\Delta t + g(x(t))\Delta\eta \tag{3.26}$$

where $\Delta\eta$ is a Gaussian variable with zero mean and variance Δt. Let η be a Gaussian process characterized by $G(t,t') = \langle \eta(t)\eta(t')\rangle = \delta(t-t')$, then one can write that $\Delta\eta = \int_t^{t+\Delta t}\mathrm{d}s\eta(s)$, as these quantities have the same statistics. They are both Gaussian and can be seen to have the same variance:

$$\langle \varDelta\eta^2 \rangle = \int_t^{t+\varDelta t} \mathrm{d}_1 \mathrm{d}t_2 \langle \eta(t_1)\eta(t_2)\rangle = \varDelta t \tag{3.27}$$

It is then natural to adopt the following continuum time formulation for Eq. (3.23)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t)) + g(x(t))\eta(t) \tag{3.28}$$

We immediately see that while it is fine to assume that the sequence of the x_n's can help us build a continuous function $x(t)$, it is certainly not right to assume that x is differentiable, simply because

$$\frac{x_{m+1} - x_m}{\varDelta t} = f(x_m) + g(x_m)\frac{\eta_m}{\sqrt{\varDelta t}} \tag{3.29}$$

and since η_m is $O(1)$ in $\varDelta t$, the last term diverges to infinity as $\varDelta t^{-1/2}$. It is therefore unlikely that Eq. (3.28) will ever appear as such in the mathematical literature. What is actually written is

$$\mathrm{d}x = f(x(t))\mathrm{d}t + g(x(t))d\eta(t) \tag{3.30}$$

where $\mathrm{d}\eta$ is a Gaussian variable with variance $\mathrm{d}t$ ($\mathrm{d}B$ or $\mathrm{d}W$ are often favorite notations among mathematicians, not $\mathrm{d}\eta$). Of course, there are anavoidable problems that we will run into by insisting to work with Eq. (3.28) as if x were a differentiable function. All these problems disappear when a discretized formulation is used.

One such problem is the following. Consider a discretized

$$\varDelta x = x(t + \varDelta t) - x(t) = f(x + \alpha\varDelta x)\varDelta t + g(x + \alpha\varDelta x)\varDelta\eta \tag{3.31}$$

where $0 \le \alpha \le 1$ is some arbitrary real number. Quite naïvely, the continuum limit of Eq. (3.31) is exactly the same as that of Eq. (3.23), namely $\dot{x} = f + g\eta$. However, as wee see from Eq. (3.31), $\varDelta x$ is $O(\sqrt{\varDelta t})$ and thus, when we write $g(x + \alpha\varDelta)\varDelta\eta$ instead of $g(x)\varDelta\eta$ we neglect terms that are of order $\alpha g'(x)\varDelta x\varDelta\eta \sim O(\varDelta t)$, that is terms of the same order as the deterministic contribution $f\varDelta t$. However, whether we write $f(x + \alpha\varDelta x)\varDelta t$ or $f(x)\varDelta t$ shouldn't make any difference in the $\varDelta t \to 0$ limit. For now, we shall stick to the discretized process defined by Eq. (3.23) or Eq. (3.26). We will now determine the statistical properties of $\varDelta x$.

3.2.2 Infinitesimal jumps

Start at time t_0 with $x = x(t_0)$ which is fixed, and consider the random variable $\varDelta x = x(t_0 + \varDelta t) - x(t_0)$ for small $\varDelta t$. We know that

$$\varDelta x = f(x(t_0))\varDelta t + g(x(t_0))\varDelta\eta \tag{3.32}$$

which is another way of rewriting $x_{m+1} - x_m = f(x_m)\varDelta t + g(x_m)\sqrt{\varDelta t}\xi_m$,. We now take the average, which leads to $\langle \varDelta x \rangle = f(x(t_0))\varDelta t$ (remember that x_0 and thus $f(x(t_0))$ are not random, since x_0 is fixed and we average only over what occurs between t_0 and $t_0 + \Delta t$). The second moment reads

$$\begin{aligned}\langle \Delta x^2 \rangle =& \langle f(x(t_0))^2 \Delta t^2 + g(x(t_0))^2 \Delta\eta^2 + 2f(x(t_0))\Delta t g(x(t_0))\Delta\eta \rangle \\ =& g(x(t_0))^2 \langle \Delta\eta^2 \rangle + O(\Delta t^{3/2}) \\ =& g(x(t_0))^2 \Delta t + O(\Delta t^{3/2})\end{aligned} \tag{3.33}$$

The third moment reads

$$\begin{aligned}\langle \Delta x^3 \rangle =& \langle f(x(t_0))^3 \Delta t^3 + g(x(t_0))^3 \Delta\eta^3 + 3f(x(t_0))\Delta t g(x(t_0))^2 \Delta\eta^2 + 3f(x(t_0))^2 \Delta t^2 g(x(t_0))\Delta\eta \rangle \\ =& O(\Delta t^{3/2})\end{aligned} \tag{3.34}$$

and we therefore realize that

$$\lim_{\Delta t \to 0} \frac{\langle \Delta x \rangle}{\Delta t} = f(x(t_0)), \quad \lim_{\Delta t \to 0} \frac{\langle \Delta x^2 \rangle}{\Delta t} = g(x(t_0))^2, \quad \lim_{\Delta t \to 0} \frac{\langle \Delta x^k \rangle}{\Delta t} = 0 \text{ for } k \geq 3 \tag{3.35}$$

This means that the process $x(t)$ is entirely defined by the first two moments of Δx as $\Delta t \to 0$.

3.2.3 Stochastic calculus: differentiation and Itō's lemma

Let's get back to the issue that we brushed upon higher up. Since there is a variety of discretized processes that naively lead to the same visual stochastic differential equation, it may be worth exploring these various discretized equations. Take $0 \leq \alpha \leq 1$ and consider Eq (3.31) which we repeat here:

$$\Delta x = x(t + \Delta t) - x(t) = f(x + \alpha \Delta x)\Delta t + g(x + \alpha \Delta x)\Delta\eta \tag{3.36}$$

For $\alpha = 0$ this is the case considered before and the Langevin equation $\dot{x} = f + g\eta$ understood in this very discretization scheme is called an Itō-discretized Langevin equation. For $\alpha = 1/2$ this is the Stratonovich discretization, and for $\alpha = 1$ this is the Hänggi-Klimontovich discretization. Before we argue about why these other discretization schemes are of any interest, we want to point that for $\alpha = 0$, the discretized Langevin equation is rather easy to implement numerically, as x_{m+1} is explicitly given in terms of x_m and of the noise ξ_m : $x_{m+1} = x_m + \Delta t f(x_m) + \sqrt{\Delta t} g(x_m)\eta_m$. Whenever $\alpha \neq 0$ this becomes an implicit equation for x_m and this is obviously less convenient, at least numerically. There are however connections between $\alpha = 0$ and $\alpha \neq 0$. To make this connection explicit, we evaluate the moments of $\Delta x = x_{m+1} - x_m = x(t+\Delta t) - x(t)$ to leading order in the $\Delta t \to 0$ limit. The first one is the trickiest one:

$$\begin{aligned}\langle \Delta x \rangle =& \langle \Delta t f(x(t) + \alpha \Delta x) + g(x(t) + \alpha \Delta x)\Delta\eta \rangle \\ =& \Delta t f(x(t)) + O(\Delta t^{3/2}) + \langle \left[g(x(t)) + \alpha g'(x(t))\Delta x \right] \Delta\eta \rangle \\ =& \Delta t f(x(t)) + O(\Delta t^{3/2}) + \alpha g'(x(t)) \langle \Delta x \Delta\eta \rangle\end{aligned} \tag{3.37}$$

and again

$$\begin{aligned}\langle \Delta x \Delta\eta \rangle =& \langle \Delta\eta \left[\Delta t f(x(t) + \alpha \Delta x) + g(x(t) + \alpha \Delta x)\Delta\eta \right] \rangle \\ =& \langle \Delta\eta g(x(t))\Delta\eta] \rangle + O(\Delta t^{3/2}) \\ =& g(x(t))\Delta t + O(\Delta t^{3/2})\end{aligned} \tag{3.38}$$

so that eventually

$$\langle \Delta x\rangle = \Delta t\left[f(x(t)) + \alpha g'(x(t))g(x(t))\right] + O(\Delta t^{3/2}) \tag{3.39}$$

By a similar method, one sees that

$$\lim_{\Delta t\to 0}\frac{\langle \Delta x^2\rangle}{\Delta t} = g(x)^2,\ \ \lim_{\Delta t\to 0}\frac{\langle \Delta x^k\rangle}{\Delta t} = 0 \text{ for } k\geq 3 \tag{3.40}$$

so that we now have a proof that whether $\dot{x} = f + g\eta$ is understood with $\alpha = 0$ or $\alpha\neq 0$, this corresponds to different physical processes because the process of Eq. (3.31) has $\frac{\langle \Delta x\rangle}{\Delta t} = f+\alpha g' g$ and $\frac{\langle \Delta x^2\rangle}{\Delta t} = g^2$. From now on, because otherwise we have no idea how to understand the corresponding Langevin equation, instead of writing $\dot{x} = f + g\eta$ we shall write $\dot{x} \stackrel{\alpha}{=} f + g\eta$ to refer to a Langevin equation that is the $\Delta t\to 0$ limit of Eq. (3.31).

That also means that it is always possible to go back and forth between two processes that are discretized differently:

$$\dot{x} \stackrel{\alpha}{=} f + g\eta \Longleftrightarrow \dot{x} \stackrel{0}{=} f + \alpha g' g + g\eta \tag{3.41}$$

or, even more generally,

$$\dot{x} \stackrel{\alpha}{=} f + g\eta \Longleftrightarrow \dot{x} \stackrel{\alpha'}{=} f + (\alpha-\alpha')g' g + g\eta \tag{3.42}$$

Henceforth, whenever a stochastic differential equation is written, and when g is not a constant, we'll specify its underlying discretization rule by dressing the equal sign with a superscript α. It would be tempting to conclude that these difficulties only occur when g is not a constant (because if g is a constant, $g' = 0$ and all the trouble disappears), and since Langevin equations with a nontrivial g are probably an exception, there is no point in spending time on these mathematical details. None of these two reasons is true: first, if x evolves according to an additive Langevin equation (that is with a constant g) then any $u(t) = U(x(t))$ won't, and second, there are plenty of situations where a multiplicative noise (g a nontrivial function of x) shows up.

In order to find out the limitations of manipulating x as if it were differentiable, we now investigate the fate of the chain rule. Consider now an auxilary random process $u(t) = U(x(t))$ built directly from x (where U is some smooth enough function). Let's see how u evolves in time:

$$\begin{aligned}
\Delta u =& u(t+\Delta t) - u(t) = U(x+\Delta x) - U(x)\\
=&\Delta x U' + \frac{1}{2}\Delta x^2 U''(x) + O(\Delta t^{3/2})\\
=&(f(x)\Delta t + g(x+\alpha\Delta x)\Delta\eta)\, U'(x) + \frac{1}{2}\Delta x^2 U''(x)\\
=&(f(x)\Delta t + \alpha g'(x)\Delta x\Delta\eta)\, U'(x) + \frac{1}{2}\Delta x^2 U''(x) + U'(x)g(x)\Delta\eta\\
=&(f(x)\Delta t + \alpha g'(x)g(x)\Delta\eta^2)\, U'(x) + \frac{1}{2}\Delta x^2 U''(x) + U'(x)g(x)\Delta\eta
\end{aligned} \tag{3.43}$$

so that

$$\lim_{\Delta t\to 0}\frac{\langle \Delta u\rangle}{\Delta t}=(f+\alpha g'g)\,U'+\frac{1}{2}g^2\,U'' \tag{3.44}$$

However, the second moment of Δu is much simpler to derive,

$$\lim_{\Delta t\to 0}\frac{\langle \Delta u^2\rangle}{\Delta t}=g^2\,U'^2 \tag{3.45}$$

and as expected $\lim_{\Delta t\to 0}\frac{\langle \Delta u^k\rangle}{\Delta t}=0$ for $k\geq 3$. This means that u too evolves according to a Langevin equation.

In the particular case $\alpha=0$, because we have

$$\text{for } \alpha=0,\ \lim_{\Delta t\to 0}\frac{\langle \Delta u\rangle}{\Delta t}=fU'+\frac{1}{2}g^2\,U'',\ \lim_{\Delta t\to 0}\frac{\langle \Delta u^2\rangle}{\Delta t}=g^2\,U'^2 \tag{3.46}$$

we see that the process must evolve according to the following Itō-discretized Langevin equation

$$\dot u\overset{0}{=}fU'+\frac{1}{2}g^2\,U''+U'g\eta \tag{3.47}$$

This Eq. (3.47) is the celebrated Itō's lemma, where the piece that corrects the standard chain rule of differential calculus has been highlighted in red.

Note also that for α arbitrary, we have that

$$\dot x\overset{\alpha}{=}f+g\eta\Longleftrightarrow \dot u\overset{0}{=}fU'+\alpha g'gU'+\frac{1}{2}g^2\,U''+U'g\eta \tag{3.48}$$

and thus, using the correspondence in Eq. (3.42)

$$\dot x\overset{\alpha}{=}f+g\eta\Longleftrightarrow \dot u\overset{\alpha}{=}fU'+\alpha g'gU'-\alpha\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(gU'\right)\right]gU'+\frac{1}{2}g^2\,U''+U'g\eta \tag{3.49}$$

or, equivalently, using that $\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(gU'\right)\right]gU'=g'gU'+g^2\,U''$

$$\dot x\overset{\alpha}{=}f+g\eta\Longleftrightarrow \dot u\overset{\alpha}{=}fU'+\alpha g'gU'+\left(\frac{1}{2}-\alpha\right)g'gU'+\left(\frac{1}{2}-\alpha\right))g^2\,U''+U'g\eta \tag{3.50}$$

This means in particular that for the Stratonovich discretization with $\alpha=1/2$ we have

$$\dot x\overset{1/2}{=}f+g\eta\Longleftrightarrow \dot u\overset{1/2}{=}fU'+U'g\eta,\ u(t)=U(x(t)) \tag{3.51}$$

That is really remarkable! This means that even though neither x nor u are differentiable, when resorting to the Stratonovich discretization, we can manipulate these functions as if they were actually differentiable since the chain rule of differentiation applies. This is not the only advantage of the Stratonovich discretization. It is also, quite simply, the only natural discretization scheme that follows from the elimination of the fast degrees of freedom.

As a conclusion to this section, we return to our original example of subsection 3.1.2 and Eq. (3.22). There we looked at $\dot{x} = \sqrt{2D}\eta$ and defined $y = x^2/2$ and without really thinking we arrive at $\dot{y} = 2\sqrt{D}\sqrt{y}\eta$. We are now more educated than back then and here is a line of reasoning within which this is correct and justified. Start from $\dot{x} = \sqrt{2D}\eta$, which can be understood in any sense one wishes, because the noise is additive (the corresponding g is a constant). For instance we can think of this equation as $\dot{x} \overset{1/2}{=} \sqrt{2D}\eta$. Then we know that the chain rule applies (this is Stratonovich discretization), and thus what we should really have written is $\dot{y} \overset{1/2}{=} 2\sqrt{D}\sqrt{y}\eta$. Using the correspondence of Eq. (3.42), we see than a strictly equivalent equation is

$$\dot{y} \overset{0}{=} \frac{1}{2}2\sqrt{D}\sqrt{y}\frac{\mathrm{d}(2D\sqrt{y})}{\mathrm{d}y} + 2\sqrt{D}\sqrt{y}\eta \overset{0}{=} D + 2\sqrt{D}\sqrt{y}\eta \tag{3.52}$$

so that $\langle\dot{y} = D$. An alternative would have been to directly use Itō's lemma to $u(t) = y(t) = x(t)^2/2$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} \overset{0}{=} U'\dot{x} + \frac{1}{2}g^2U'' + gU'\eta \overset{0}{=} x(\sqrt{2D}\eta) + \frac{1}{2}\times 2D\times 1 + \sqrt{2D}x\eta \overset{0}{==} D + 2\sqrt{Dy}\eta \tag{3.53}$$

which is consistent.

3.3 Why bother, in physics, about multiplicative noise?

It's time that we connect back the abstract Langevin equation $\dot{x} = f + g\eta$ to some actual physical processes. We want to identify the degrees of freedom x that evolves according to a stochastic differential equation (x will be shown to be either the velocity or the position). We want to identify what the deterministic contribution f is in relevant physical situations. And finally, we have seen that working with a nontrivial function g (that is not a constant) leads to a host of mathematical difficulties. Was it really worth the pain if no physical situations with a nontrivial g can be encountered? We'll see that unfortunately, such nontrivial functions g appear everywhere.

3.3.1 Starting with an additive noise

Start with the example of the colloid in water, without any external forces

$$m\dot{v} = -\gamma v + \sqrt{2\gamma' T}\eta \tag{3.54}$$

in which the discretization is irrelevant. If interested in the evolution of the kinetic energy $K = mv^2/2$ of the particle (in one space dimension, for simplicity) we can write

$$\dot{K} = -\gamma\frac{2}{m}K + \sqrt{2\gamma' T}\sqrt{\frac{2K}{m}}\eta \tag{3.55}$$

but the resulting equation does feature a multiplicative noise. Since the discretization is irrelevant, let's assume in the first place that it was Stratonovich discretized, because then we know that we can manipulate $v(t)$ as if it were a differentiable function, and thus the resulting equation for K is also Stratonovich-discretized, and we should write

$$\dot{K} \overset{1/2}{=} -\gamma\frac{2}{m}K + 2\sqrt{\frac{\gamma' TK}{m}}\eta \tag{3.56}$$

which is equivalent to

$$\begin{aligned}\dot{K} &\overset{0}{=} -\gamma\frac{2}{m}K + \frac{1}{2}\left[2\sqrt{\frac{\gamma' TK}{m}}\right]\frac{\mathrm{d}}{\mathrm{d}K}\left[2\sqrt{\frac{\gamma' TK}{m}}\right] + 2\sqrt{\frac{\gamma' TK}{m}}\eta \\ &\overset{0}{=} -\gamma\frac{2}{m}K + \frac{\gamma' T}{m} + 2\sqrt{\frac{\gamma' TK}{m}}\eta\end{aligned} \tag{3.57}$$

and thus $\langle\dot{K}\rangle = \gamma\frac{2}{m}\left(\langle K\rangle - \frac{\gamma'}{\gamma}T/2\right)$, which confirms that in equilibrium equipartition is achieved on condition that $\gamma' = \gamma$. Hence, the dynamics of the colloid is constrained by the fact that at large times it is in thermal equilibrium. This imposes $\gamma = \gamma'$. The conclusion is that even if we started off from an additive Langevin equation (g is a constant) the Langevin equation for K has a multiplicative noise (g is not a constant).

Note that if an external force field was acting on the particle we would have

$$m\dot{v} = -\gamma v - V'(x) + \sqrt{2\gamma T}\eta \tag{3.58}$$

If the viscous damping is large, this means that the force $F = -V'$ very rapidly balances out the force exerted by the thermostat, $F_b = -\gamma v + \sqrt{2\gamma T}\eta$, and the inertial term $m\dot{v}$ can be omitted. When this overdamped limit is justified, the Langevin equation takes the form

$$\gamma\dot{x} = -V' + \sqrt{2\gamma T} \text{ or } \dot{x} = -\mu V'(x) + \sqrt{2\mu T}\eta \tag{3.59}$$

where $\mu = \gamma^{-1}$ is the so-called mobility. When in addition $V(x) = \frac{k}{2}x^2$ is a harmonic potential, then the resulting Langevin equation is exactly of the same form as Eq. (3.54), $\dot{x} = -\mu kx + \sqrt{2\mu T}\eta$. A Langevin equation of the form $\dot{y} = -\kappa y + g\eta$, where g is a constant, is an Ornstein-Uhlenbeck process. It is the only stationary Gaussian process (by Doob's theorem). It is extremely important to fully master the details of the properties of an Ornstein-Uhlenbeck process (hence its repeated appearances in the tutorials).

3.3.2 Plain diffusion, confined though

The viscous friction γ is in general a constant coefficient, except when the vicinity of a wall affects the hydrodynamic flow around the particle under consideration. With the upsurge of microfluidic devices, diffusion in confined geometries (micro and nanofluidics, pores, channels, even in quasi 1d settings) demands that hydrodynamic interactions be taken into consideration. Some recent experimental references are [68, 1, 45, 96]. the bottom line is that if z denotes the distance of the particle to the wall, the transverse and longitudinal friction coefficients pick up a dependence on z

$$\gamma_\perp(z) = \gamma\left(1+\frac{9}{8}\frac{R}{z}\right) \tag{3.60}$$

$$\gamma_\parallel(z) = \gamma\left(1+\frac{9}{16}\frac{R}{z}\right) \tag{3.61}$$

as derived by Brenner [13]. This means that even when describing the purely diffusive motion of a confined colloid, multiplicative noise will show up.

3.3.3 Black and Scholes

Black and Scholes were two scholars working in on the modeling of financial markets. Back in 1973 [9] they came up with a model for the evolution of the price of a specific type of asset (European-style option). We refer to [30] for an introduction and for financial motivations. The equation they wrote is a Langevin equation for the shrare price S of a stock (a risky asset). In Itō discretization, it is postulated that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu S + \sigma S\eta \tag{3.62}$$

where η has unit variance and where σ is the volatility of the stock. In the Stratonovitch discretization we have

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mu S - \frac{1}{2}\sigma^2 S + \sigma S\eta \tag{3.63}$$

so that $S(t) = S_0 \exp\left[(\mu - \sigma^2/2)t + \sigma\int_0^t \mathrm{d}\tau\eta(\tau)\right]$. Of course μ could also depend on S or on time (and then things would have to be changed). One ends up facing the statistics of an exponential functional of Brownian motion. This type of functionals has appeared in mathematical finance [121], but also in the physics of one-dimensional disordered systems [22], or in the nonequilibrium evolution of chemical processes [48]. The Black and Scholes equation is a prototypical example of a Langevin equation with multiplicative noise.

3.3.4 Rotational diffusion

Dielectric relaxation. Let $\mathbf{p}$ be an electric dipole, such that $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \boldsymbol{\omega}\times\mathbf{p}$ with the equation of motion $I\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = -\zeta\boldsymbol{\omega} + \mathbf{p}\times\mathbf{E} + \boldsymbol{\lambda}$. Neglecting inertia, one ends up with $\zeta\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = (\mathbf{p}\times\mathbf{E})\times\mathbf{p} + \boldsymbol{\lambda}\times\mathbf{p}$. Multplicative noise occurs and the Stratonovich rule is to be understood. It is the only one consistent with modulus conservation. If in a simulation the Itô rule is erroneously used, then modulus stops being conserved.

3.3.5 Active Brownian Particle

When one models the motion of an active particle under the action of a self-propulsion force, it turns out that writing

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = v_0\mathbf{u} \tag{3.64}$$

faithfully reflects the observed properties of an individual particle [18]. In this model, $\mathbf{u}$ is a unit vector the tip of which executes a Brownian motion at the surface of the unit sphere, with a rotational diffusion constant D_r. In two space dimensions, the polar angle ϕ indexing the direction of $\mathbf{u} = (\cos\phi, \sin\phi)$ is undergoing a simple Brownian motion,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \sqrt{2D_r}\xi \tag{3.65}$$

where ξ is a Gaussian white noise with correlations $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. But these particles live in three dimensions, and $\mathbf{u}$ is characterized by two angles. The evolution equation for $\mathbf{u}$ reads

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \overset{\alpha}{=} (\mathbf{u}\cdot\boldsymbol{\eta})\mathbf{u} - \mathbf{u}^2\boldsymbol{\eta} \tag{3.66}$$

where the components η^μ of $\boldsymbol{\eta}$ are independent Gaussian white noises, $\langle \eta^\mu(t)\eta^\nu(t')\rangle = 2D_r\delta^{\mu\nu}\delta(t-t')$. Dotting Eq. (3.67) with $\mathbf{u}$ shows that $\mathbf{u}^2$ is indeed conserved. But for this calculation to be legitimate, we must be allowed to manipulate $\mathbf{u}$ as if it were a smooth function, which means that Eq. (3.67) must be understood in a Stratonovich sense ($\alpha = 1/2$). It's Itō counterpart reads

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \overset{0}{=} -(d-1)D_r\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\eta})\mathbf{u} - \mathbf{u}^2\boldsymbol{\eta} \tag{3.67}$$

where d is the number of space dimensions.

Formula Eq. (3.67) can checked using Eq. (**??**), with $g^{\mu i} = \sqrt{2D_r}(u^\mu u^i - \mathbf{u}^2\delta^{\mu i})$ (here $i, \mu = 1, \dots, d$), we have $\partial_\nu g^{\mu i} = \sqrt{2D_r}(\delta^{\mu\nu}u^i + u^\mu\delta^{\nu i} - 2u^\nu\delta^{\mu i})$, and thus, after an explicit evaluation,

$$\partial_\nu g^{\mu i} g^{\nu i} = -(d-1)2D_r u^\mu \mathbf{u}^2 \tag{3.68}$$

so that $a_1^\mu = \frac{1}{2}\partial_\nu g^{\mu i} g^{\nu i} = -(d-1)D_r u^\mu \mathbf{u}^2$.