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# Wilson renormalization of a reaction–diffusion process

F. van Wijland<sup>a,\*</sup>, K. Oerding<sup>b</sup>, H.J. Hilhorst<sup>a</sup><sup>a</sup> *Laboratoire de Physique Théorique et Hautes Energies Bâtiment 211, Université de Paris-Sud, 91405 Orsay cedex, France<sup>†</sup>*<sup>b</sup> *Institut für Theoretische Physik III, Heinrich Heine Universität, 40225 Düsseldorf, Germany*

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## Abstract

Healthy and sick individuals ( $A$  and  $B$  particles) diffuse independently with diffusion constants  $D_A$  and  $D_B$ . Sick individuals upon encounter infect healthy ones (at rate  $k$ ), but may also spontaneously recover (at rate  $1/\tau$ ). The propagation of the epidemic therefore couples to the fluctuations in the total population density. Global extinction occurs below a critical value  $\rho_c$  of the spatially averaged total density. The epidemic evolves as the diffusion–reaction–decay process  $A + B \rightarrow 2B$ ,  $B \rightarrow A$ , for which we write down the field theory. The stationary-state properties of this theory when  $D_A = D_B$  were obtained by Kree et al. The critical behavior for  $D_A < D_B$  is governed by a new fixed point. We calculate the critical exponents of the stationary state in an  $\epsilon$  expansion, carried out by Wilson renormalization, below the critical dimension  $d_c = 4$ . We then go on to obtain the critical initial time behavior at the extinction threshold, both for  $D_A = D_B$  and  $D_A < D_B$ . There is nonuniversal dependence on the initial particle distribution. The case  $D_A > D_B$  remains unsolved. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction and phenomenological analysis

### 1.1. Introduction

We study the propagation of an epidemic in a population of fluctuating density, with parameters set such that the epidemic is close to becoming extinct. The model is as follows. Healthy and sick individuals, also called  $A$  and  $B$  particles, diffuse freely and independently on a lattice of dimension  $d$ . Their diffusion constants  $D_A$  and  $D_B$  are

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\* Corresponding author. E-mail: fvw@stat.th.u-psud.fr.

<sup>†</sup> Laboratoire associé au Centre National de la Recherche Scientifique - URA D0063.

in general unequal. Sick individuals upon encounter infect healthy ones (at a rate  $k$ ), but may also spontaneously recover (at a rate  $1/\tau$ ). The epidemic therefore tends to propagate in regions of high total population density and to become locally extinct in regions of low density. Below a critical density  $\rho_c$  global extinction occurs, and an important question is how the epidemic evolves near this threshold, which is analogous to a phase transition point in equilibrium critical phenomena. The role of the order parameter is played by the  $B$  particle density  $\rho_B$ . In particular, at threshold strong local fluctuations are expected, with a potentially decisive impact on the global critical behavior.

The evolution of the epidemic may be formulated as the competition between a reaction and a decay process of two diffusing chemical species,



where  $A$  and  $B$  stand for a healthy and a sick individual, respectively.

Let  $m_i$  and  $n_i$  be the number of  $A$  and  $B$  particles, respectively, that occupy site  $i$ . We write  $m = \{m_i\}$  and  $n = \{n_i\}$  for the two sets of occupation numbers and denote by  $P(m, n; t)$  the probability that at time  $t$  the system is characterized by exactly these sets. The rules of the process formulated above then define a master equation for the time evolution of  $P(m, n; t)$  with fully specified transition rates. The problem depends on the parameters  $D_A$ ,  $D_B$ ,  $k$ ,  $1/\tau$ , and  $\rho$ . After scaling one may set  $k = 1$ .

This problem is but one instance of a large class of such processes that all have the following common features. They are initially formulated in terms of a master equation which is hard, if not impossible, to solve. The traditional approach is to write down rate equations for the space- and time-dependent densities of the various species. This leads to numerous interesting systems of partial differential equations (*reaction-diffusion equations*) that continue to be subjects of investigation.

The assumptions that lead to such reaction-diffusion equations can be justified only in sufficiently high spatial dimension  $d$ . The reason is that these equations constitute a *mean field* description: the reaction terms involve the products of the *average* densities, rather than the average of the product of the fluctuating densities.

The general problem of statistical mechanics is therefore to determine the critical dimension  $d = d_c$  below which the mean-field approach becomes qualitatively incorrect, and to describe the modifications that then appear. Recent progress on such problems is due to the application of the renormalization group (RG) to the formal solution of the master equation, rewritten as a path integral and cast in the form of a field theory. This will also be our procedure. It allows one, in an expansion in the small parameter  $d_c - d = \epsilon$ , to study the scale-invariant regimes of the problem and determine the associated exponents.

For the particular process of Eq. (1.1) the upper critical dimension appears to be  $d_c = 4$ . Questions of interest are first of all about the properties of the stationary state and secondly about how the system, prepared in a specified initial state, relaxes towards stationarity. Of particular interest is the initial state with the  $A$  and  $B$  particles

distributed randomly with densities  $\rho_A^0$  and  $\rho_B^0$ . In that case, and in the limit of an infinite lattice, the  $m_i$  and  $n_i$  are independent Poissonian random variables with averages  $\rho_A^0$  and  $\rho_B^0$ , respectively.

We now relate our work to existing literature. We shall arrive at a field theory described by an action  $S$  (see Eq. (1.16)) in which

$$\mu = \frac{D_A - D_B}{D_A} \quad (1.2)$$

is a key parameter. The stationary state of a special case, viz. with  $D_A = D_B$ , was studied recently by Kree et al. [1]. These authors' physical interpretation is different: They start from a pair of Langevin equations postulated for the evolution of a population density in a polluted environment. The interest of both this work and of Ref. [1] lies in the coupling between the fluctuations of the total density and those of the density of the species that is trying to survive (in our case the  $B$  particles). If the fluctuations in the total density are suppressed, the full action  $S$  becomes independent of  $\mu$  and reduces to an action  $S_{DP}$  known to represent both the problem of directed percolation and the Schlögl reaction.  $S_{DP}$  has first appeared in particle physics as Reggeon field theory.

This work goes beyond the existing results in the following ways.

(i) The ratio  $\mu$  appears to be relevant in the sense of the RG. For  $\mu < 0$  the critical behavior is governed by a new fixed point. We determine the new exponents to leading order in  $\varepsilon = 4 - d$ . For  $\mu > 0$  the RG equations do not have a fixed point and we speculate about the interpretation of this fact.

(ii) We consider relaxation to the stationary state at the critical density  $\rho = \rho_c$ , first for the action  $S_{DP}$  of directed percolation and then for the full action  $S$  both with  $\mu = 0$  and  $\mu < 0$ . For asymptotically long times the order parameter  $\rho_B(t)$  relaxes towards zero as a power law of time whose exponent is a simple combination ( $\beta/\nu$  in the usual notation) of the stationary state exponents. There is, however, an initial time regime – which in the limit  $\rho_B^0 \downarrow 0$  becomes arbitrarily long – where the order parameter increases as  $t^{\theta'}$ , where  $\theta'$  is a new and independent *critical initial slip* exponent. We calculate this exponent for an initial state in which the  $m_i$  and  $n_i$  are independent but not necessarily Poissonian random variables. We show that  $\theta'$  is *nonuniversal*, as it depends continuously on the width of the distribution. The concept of critical initial slip was introduced by Janssen et al. [2] (see also Ref. [3]) and it is here for the first time, to our knowledge, that such an exponent is calculated for a reaction-diffusion system.

(iii) Finally, we do not follow the usual field-theoretic methods. Instead, we perform a Wilson renormalization, that is, we successively integrate out shells of large momentum. We employ the dynamical version of Wilson's method developed by Hohenberg and Halperin [4]. This method does not require any advanced knowledge of field theory. It has been applied earlier by Droz and Sasvári [5] to one species annihilation reactions. In the course of our work, we check that in known special cases earlier results are reproduced.

### 1.2. Mean-field theory

The mean-field equations for the densities  $\rho_A$  and  $\rho_B$  are

$$\begin{aligned}\frac{\partial \rho_A}{\partial t} &= D_A \Delta \rho_A + \frac{1}{\tau} \rho_B - k \rho_A \rho_B, \\ \frac{\partial \rho_B}{\partial t} &= D_B \Delta \rho_B - \frac{1}{\tau} \rho_B + k \rho_A \rho_B.\end{aligned}\quad (1.3)$$

By summing them one sees that the total density  $\rho = \rho_A + \rho_B$  is conserved. The spatially homogeneous solutions of these equations are easily found. The total density  $\rho$  appears to have a threshold value  $\rho = \rho_c^{mf} = (k\tau)^{-1}$  below which the stationary state consists of only  $A$  particles. Above this threshold density the all- $A$  state is still stationary but unstable, and the stable stationary state has a nonzero density of  $B$  particles. Explicitly, Eq. (1.3) leads to

$$\begin{aligned}\rho_A(\infty) &= \rho_c^{mf}, & \rho_B(\infty) &= \rho - \rho_c^{mf} & \text{for } \rho > \rho_c^{mf}, \\ \rho_A(\infty) &= \rho, & \rho_B(\infty) &= 0 & \text{for } \rho < \rho_c^{mf}.\end{aligned}\quad (1.4)$$

If we write, in analogy with critical phenomena in thermodynamic systems,

$$\rho_B(\infty) \sim (\rho - \rho_c)^\beta \quad (1.5)$$

then one sees that mean-field theory gives  $\beta = 1$ . The relaxation to the stationary state at criticality from a homogeneous initial state with  $B$ -particle density  $\rho_B^0$  is easily derived from Eq. (1.3),

$$\rho_B(t) = \frac{\rho_B^0}{1 + \rho_B^0 k t}. \quad (1.6)$$

Again in analogy with critical relaxation in thermodynamic systems we define the exponents  $z_V$  and  $\theta'$  by

$$\rho_B(t) \sim t^{-\beta/(z_V)} \quad \text{for } t \gg (\rho_B^0 k)^{-1}, \quad (1.7)$$

$$\rho_B(t) \sim \rho_B^0 t^{\theta'} \quad \text{for } \rho_B^0 k t \ll 1. \quad (1.8)$$

Here  $\theta'$  is the *critical initial slip* exponent introduced by Janssen et al. [2,3]. Eq. (1.6) shows that the mean-field values are  $z_V = 1$  and  $\theta' = 0$ . In the following sections, after showing that the reaction–diffusion process of Eq. (1.1) has upper critical dimension  $d_c = 4$ , we will compute corrections to the mean-field values of these and other exponents to first order in  $\varepsilon = 4 - d$ .

### 1.3. The master equation solved in terms of a path integral

In order to go beyond the mean-field theory, one has to start from the master equation defined implicitly in Section 1.1. It is convenient to introduce an orthonormal basis of

states  $|m, n\rangle$  and work in the Hilbert space generated by this basis. The probability distribution  $P(m, n; t)$  is then associated with the state

$$|P(t)\rangle = \sum_{m,n} P(m, n; t) |m, n\rangle . \tag{1.9}$$

For an appropriately constructed time evolution operator  $\hat{H}$ , the master equation for  $P(m, n; t)$  is equivalent to the imaginary time Schrödinger equation

$$\frac{d}{dt} |P(t)\rangle = -\hat{H} |P(t)\rangle . \tag{1.10}$$

From the formal solution of Eq. (1.10),

$$|P(t)\rangle = e^{-\hat{H}t} |P(0)\rangle \tag{1.11}$$

all properties of the reaction–diffusion system may be derived. Eq. (1.11) determines in particular in the limit  $t \rightarrow \infty$  the stationary state. The combined efforts of many workers, e.g. [6–8], have led to methods of converting the formal solution of type, Eq. (1.11), into expressions that can be analyzed. We shall use here one such method, which has now become common. The first step is to express  $\hat{H}$  in terms of creation and annihilation operators for the two particle species:  $a_i^\dagger$  and  $a_i$  for the  $A$  particles, and  $b_i^\dagger$  and  $b_i$  for the  $B$  particles. These operators can be defined such that they have boson commutation relations. Next, the exponential in Eq. (1.11) is subjected to time slicing with the aid of a coherent state representation of the harmonic oscillator operators. As a result, and in the limit where the site index becomes a continuous space vector  $x$ , there appear space and time dependent classical fields  $a^*(x, t)$ ,  $a(x, t)$  and  $b^*(x, t)$ ,  $b(x, t)$  that are directly associated with the two particle types. The solution (Eq. (1.11)) finally takes the form of a path integral on these fields, weighted with the exponential of an action  $S$ :

$$|P(t)\rangle = \int \mathcal{D}a \mathcal{D}a^* \mathcal{D}b \mathcal{D}b^* e^{-S[a, a^*, b, b^*]} |P(0)\rangle . \tag{1.12}$$

The exact way in which such physical quantities as local densities, correlation and response functions can be obtained from the fields is detailed in Ref. [8].

#### 1.4. Action and scaling dimensionalities

The full action for the problem of Eq. (1.1) is

$$\begin{aligned} S[a, a^*, b, b^*] = & \int d^d x dt [a^*(\partial_t - D_A A)a + b^*(\partial_t - D_B B)b \\ & + kab(a^* - b^*)b^* - \frac{1}{\tau}(a^* - b^*)b] \\ & - \int d^d x [\rho_A^0 a^*(x, 0) + \rho_B^0 b^*(x, 0) + a(x, T) + b(x, T)] . \end{aligned} \tag{1.13}$$

Here  $\rho_A^0$  and  $\rho_B^0$  are the initial  $A$  and  $B$  particle densities, and Poissonian distribution of the initial occupation numbers has been assumed, and the time integral runs

through a fixed interval  $[0, T]$ . The averages with weight  $e^{-S}$  of the fields  $a(x, t)$  and  $b(x, t)$  are equal [6–8] to the local densities of the  $A$  and  $B$  particles. In order to cast Eq. (1.13) in a form suitable to subsequent analysis, we first change from  $a^*$  and  $b^*$  to the variables

$$\bar{a} = a^* - 1 \quad \text{and} \quad \bar{b} = b^* - 1 \quad (1.14)$$

the effect of which is to get rid of the boundary terms at time  $t = T$ . Then we introduce the fields

$$\begin{aligned} \varphi &= a + b - \rho \Theta(t), & \bar{\varphi} &= \bar{a}, \\ \psi &= b, & \bar{\psi} &= \bar{b} - \bar{a}. \end{aligned} \quad (1.15)$$

where the parameter  $\rho$  denotes as before the average total particle density, and  $\Theta(t)$  is the unit step function. After further rescaling of the fields and time, one finds the expression for the action that will be the starting point in the following sections:

$$\begin{aligned} S[\varphi, \bar{\varphi}, \psi, \bar{\psi}] &= \int d^d x dt [\bar{\varphi}(\partial_t - A)\varphi + \bar{\psi}(\partial_t + \lambda(\sigma - A))\psi \\ &\quad + \mu \bar{\varphi} \Delta \psi + g \psi \bar{\psi}(\psi - \bar{\psi}) + u \psi \bar{\psi}(\varphi + \bar{\varphi}) \\ &\quad + v_1(\psi \bar{\psi})^2 + v_2 \psi \bar{\psi}(\psi \bar{\varphi} - \bar{\psi} \varphi) + v_3 \varphi \bar{\varphi} \psi \bar{\psi} \\ &\quad - \rho_B^{(0)} \delta(t) \bar{\psi}]. \end{aligned} \quad (1.16)$$

In terms of the original parameters of the master equation the coupling constants of this action are given by

$$\begin{aligned} \mu &= 1 - D_B/D_A, \quad g = k \sqrt{\rho}/D_A, \quad u = -k \sqrt{\rho}/D_A, \\ v_1 = v_2 = -v_3 &= k/D_A, \quad \lambda \sigma = k(\rho_c^{mf} - \rho)/D_A, \quad \lambda = D_B/D_A, \\ \rho_B^{(0)} &= \rho_B^0 / \sqrt{\rho}. \end{aligned} \quad (1.17)$$

If one omits from the action Eq. (1.16) the initial time term proportional to  $\rho_B^{(0)}$ , then the remainder is, for  $\mu = 0$ , invariant under the time-reversal symmetry

$$\begin{aligned} \psi(x, t) &\rightarrow -\bar{\psi}(x, -t), \\ \bar{\psi}(x, t) &\rightarrow -\psi(x, -t), \\ \varphi(x, t) &\rightarrow \bar{\varphi}(x, -t), \\ \bar{\varphi}(x, t) &\rightarrow \varphi(x, -t). \end{aligned} \quad (1.18)$$

The breaking of this symmetry for  $\mu \neq 0$ , i.e. when the diffusion constants  $D_A$  and  $D_B$  are different, will be seen to have far-reaching consequences for the critical behavior of this system. The naive scaling dimensionalities of the coupling constants are, in powers of an inverse length,

$$[g] = [u] = 2 - d/2, \quad [v_i] = 2 - d, \quad [\mu] = 0. \quad (1.19)$$

This shows that  $d_c = 4$  is the upper critical dimension for this problem and that in an expansion around  $d = 4$  we may drop the  $v_i$  vertices.

The terms occurring in Eq. (1.16) may be given the following physical interpretation. The  $g$  vertex, which involves only the  $B$  fields  $\psi$  and  $\bar{\psi}$ , accounts for fluctuations in the contamination process  $A + B \rightarrow 2B$  due to fluctuations in the  $B$  particle density, and as though the  $A$  particles formed a homogeneous background. The sum  $\varphi + \bar{\varphi}$  may be shown to be equal to the fluctuating part of the total density. The  $u$  vertex couples this to the density of  $B$  particles, expressing that the reaction  $A + B \rightarrow 2B$  is also affected by fluctuations of the  $A$  particle density.

### 1.5. A reduced problem: Directed percolation

If in the action  $S$  of Eq. (1.16) we replace the fields  $\varphi$  and  $\bar{\varphi}$  by their mean-field values  $\varphi = 0$  and  $\bar{\varphi} = 0$ , and drop the fourth-order terms that were argued above to be irrelevant, the result is the action  $S_{DP}$  known in particle physics as the *Reggeon field theory* and given by

$$S_{DP}[\psi, \bar{\psi}] = \int d^d x dt \left[ \bar{\psi}(\partial_t + \lambda(\sigma - \Delta))\psi + g\psi\bar{\psi}(\psi - \bar{\psi}) - \rho_B^{(0)} \delta(t)\bar{\psi} \right]. \quad (1.20)$$

Physically, this means that we have constrained the total density to be constant (equal to  $\rho$ ) in space and time. If the  $B$  density is much lower than the  $A$  density, as is the case near criticality, this constraint means that the  $B$  particles interact with a continuous background. The first link between the action  $S_{DP}$  and statistical mechanics was established by Cardy and Sugar [9], who showed that it describes the problem of directed bond percolation. Subsequently, it was shown by Grassberger and Sundermeyer [10] and by Grassberger and de la Torre [11] (see also Ref. [12]) that this action also describes Schlögl's chemical reaction [13]. The  $\varepsilon$  expansion for the exponents of the action  $S_{DP}$  was known from field theory (Ref. [14] and references therein) and has been extended by Janssen [12].

The remainder of this article is organized as follows. In Section 2, for a twofold reason, we return to the action  $S_{DP}$ . We consider for the first time its relaxation from an initial state and calculate the initial critical slip exponent  $\theta'$ . Secondly, since throughout this work we employ time-dependent Wilson renormalization, we explain our method on the example of the relatively simple action  $S_{DP}$ , after which we apply it with only few extra comments to the full action  $S$  of Eq. (1.16).

In Section 3 we return to the full action  $S$ . We determine its stationary-state properties, which are then used to study the relaxation from the initial state. We also study the effect of non-Poissonian initial particle distributions.

## 2. Directed percolation: Wilson renormalization and critical initial slip exponent

The stationary state of the action Eq. (1.20) exhibits critical behavior that has been studied by several authors both within the framework of an  $\varepsilon$ -expansion around its upper

critical dimension  $d = 4$ . The analysis of the action Eq. (1.20) is usually performed (e.g. Refs. [9,12]) by means of field-theoretic methods, based upon the perturbative expansion of vertex functions and dimensional regularization, and leading to Callan–Symanzik equations. In this work, instead, we apply Wilson’s dynamic renormalization group, i.e. successively eliminate short wavelength degrees of freedom. This method has already been used successfully to study the dynamics of spin systems for instance; it is applied here for the first time to a reaction–diffusion problem.

### 2.1. Wilson’s dynamic renormalization group

The extension of Wilson’s renormalization procedure to time-dependent systems has been described by Hohenberg and Halperin [4]. We recall here how this method works on the example of the action  $S_{DP}$  of Eq. (1.20). For a study restricted to the stationary state, the boundary term at  $t = 0$  may be suppressed. The action may then be written as  $S_{DP} = S_0 + S_{int}$  with the free term  $S_0$  given by

$$S_0[\psi, \bar{\psi}] = \int \bar{\psi}(\partial_t + \lambda(\sigma - A))\psi. \quad (2.1)$$

The free propagator, in terms of the spatial Fourier components, is

$$\langle \bar{\psi}(-k, t')\psi(k, t) \rangle = \Theta(t - t')e^{-(k^2 + \sigma)(t - t')}. \quad (2.2)$$

One is interested in averages carried out with the weight  $e^{-S_{DP}}$ . In Wilson’s procedure, this weight is first integrated over the Fourier components of the fields with wave vectors  $k$  in a momentum shell  $\Omega_A$  defined by

$$\Omega_A = \{k \mid A/b < k < A\}. \quad (2.3)$$

We denote these fields for brevity by  $\psi_>$  and  $\bar{\psi}_>$  and the remaining fields by  $\psi_<$  and  $\bar{\psi}_<$ . The scale factor  $b$  will be taken infinitesimally close to 1, and the momentum space cut-off  $A$  is of order unity. One obtains an effective action  $S_{DP}^{eff}$

$$e^{-S_{DP}^{eff}[\psi_<, \bar{\psi}_<]} = \int \mathcal{D}[\psi_>, \bar{\psi}_>] e^{-S_{DP}}. \quad (2.4)$$

In practice, the integration step is carried out perturbatively in  $S_{int}$  by using the cumulant expansion

$$\exp(-S_{DP}^{eff}) = \exp(-\langle S_{int} \rangle' + \frac{1}{2!} \langle S_{int}^2 \rangle'_c - \frac{1}{3!} \langle S_{int}^3 \rangle'_c + \dots). \quad (2.5)$$

The index  $c$  indicates a cumulant and  $\langle \dots \rangle'$  stands for an average on the  $>$  fields with weight  $e^{-S_0}$ . Therefore, there remain only Gaussian integrations to perform.

Wick’s theorem ensures that the integrations reduce to pair contractions, and the graphical rules for dealing with these are easily found. It is convenient (see Fig. 1) to represent a  $\bar{\psi}$  field (a  $\psi$  field) by a leg with (without) an arrow. The interaction term  $S_{int}$  is a sum of three-leg vertices of types  $g_1$  and  $g_2$  in Fig. 1. The  $n$ th-order



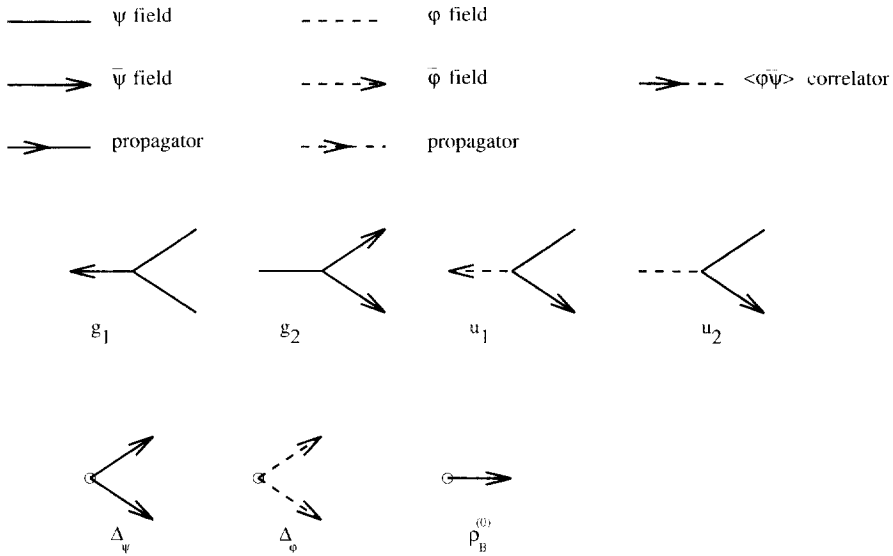


Fig. 1. Graphical conventions. The  $\psi$  propagator is shown in plain line, the  $\phi$  propagator appears as a dashed line, and whenever it is non-zero, the  $\langle \bar{\psi}\psi \rangle$  correlator connects a plain leg to a dashed one. Also shown are the vertices comprising the interaction term of the full action. A vertex connecting legs by an empty circle  $\circ$  is attached to  $t=0$  and is therefore not summed over time.

cumulant in Eq. (2.5) comes from diagrams with  $n$  such vertices. An external leg of a diagram will stand for a  $\langle$  field and an internal line for the contraction of two  $\rangle$  fields. Each vertex is integrated over space and time. Only one-particle irreducible diagrams need to be considered. Moreover, any diagram in which one can go around a loop in the direction of the arrows vanishes by causality. A contraction of any pair of fields at equal times is zero, a property conveniently expressed by the convention  $\Theta(0)=0$  and commented upon, e.g. in Ref. [3]. Finally, products of noncontracted fields at different times are reduced to single-time expressions by Taylor expansion in the time difference.

After the integration on the momentum shell fields is done one applies to  $S_{DP}^{eff}$  a scale transformation which depends on exponents  $z, d_\psi$ , and  $d_{\bar{\psi}}$  still to be determined,

$$\begin{aligned} x &= bx', & \psi_{<}(x, t) &= b^{-d_\psi} \psi'(x', t'), \\ t &= b^z t', & \bar{\psi}_{<}(x, t) &= b^{-d_{\bar{\psi}}} \bar{\psi}'(x', t'). \end{aligned} \tag{2.6}$$

and drops the primes of the new variables. The result is a renormalized action whose coupling constants are given in terms of the original couplings. In dimension  $d = d_c - \epsilon$ , the cumulant series in Eq. (2.5) usually corresponds to an expansion in powers of  $\epsilon$  and upon truncating it at the desired order, one has a RG involving only a finite number of coupling constants. If a fixed point of the transformation can be found, then scale invariance is a direct consequence and scaling laws follow.

## 2.2. Stationary state of $S_{DP}$

The time-reversal symmetry of Eq. (1.18) imposes that  $d_\psi = d_{\bar{\psi}}$ . We write  $d_\psi = d_{\bar{\psi}} = \frac{1}{2}(d + \eta)$ , where  $\eta$  is the anomalous dimension of the fields  $\psi$  and  $\bar{\psi}$ . We have applied the renormalization procedure of Section 2.1 to  $S_{DP}$ , taking into account all diagrams with two and three vertices. The resulting recursion relations read, with  $b \equiv e'$ ,

$$\frac{dg}{d\ell} = g \left[ z - 2 + \frac{\varepsilon}{2} - \frac{3\eta}{2} - 2g^2 \frac{K_4}{\lambda^2} \right] \quad (2.7)$$

for the coupling constant of the three-leg vertex, and

$$\frac{d\sigma}{d\ell} = (z - \eta)\sigma + \frac{g^2 K_4}{\lambda} \frac{A^4}{A^2 + \sigma} \quad (2.8)$$

for the deviation from the critical density. Here  $K_4$  is the surface area of the unit sphere in  $\mathbb{R}^4$  divided by  $(2\pi)^4$ . Requiring that the coefficients of the terms with the time and space derivatives in  $S_0$  remain constant leads to the conditions

$$-\eta - g^2 \frac{K_4}{2\lambda^2} = 0, \quad (2.9)$$

$$z - 2 - \eta - g^2 \frac{K_4}{4\lambda^2} = 0. \quad (2.10)$$

These allow one to express the exponents  $z$  and  $\eta$  in terms of  $g^2 \lambda^{-2}$ . Eqs. (2.7) and (2.10) have a nontrivial fixed point,

$$g^{\hat{\varepsilon}2} = \frac{\lambda^2}{3K_4} \varepsilon, \quad (2.11)$$

where we neglect terms of higher order in  $\varepsilon$ . This fixed-point is stable since in its vicinity  $g$  scales with the negative exponent  $y_g = -\varepsilon$ . The fixed-point value of  $\sigma$  is

$$\sigma^{\hat{\varepsilon}} = -\frac{\lambda A^2}{6} \varepsilon. \quad (2.12)$$

In view of the definition of  $\sigma$  this implies that the critical density  $\rho_c$  in dimension  $4 - \varepsilon$  is higher than the mean field value  $\rho_c^{mf} = 1/k\tau$ . Near the fixed point  $\sigma$  and time scale with the exponents

$$y_\sigma =: \nu^{-1} = 2 - \frac{\varepsilon}{4}, \quad (2.13)$$

$$z = 2 - \frac{\varepsilon}{12}, \quad (2.14)$$

respectively, and the anomalous dimension of the fields  $\psi$  and  $\bar{\psi}$  is

$$\eta = -\frac{\varepsilon}{6}. \quad (2.15)$$

In Eq. (2.13),  $\nu$  is the usual correlation length exponent. Since  $\beta = \frac{1}{2}\nu(d + \eta)$ , the density of the  $B$  species scales with

$$\beta = 1 - \frac{\varepsilon}{6}. \tag{2.16}$$

The analysis has been extended to second order in  $\varepsilon$  by Janssen [12].

The logarithmic correction in dimension  $d = 4$ ,

$$\rho_B(\infty) \sim |\sigma| \ln^{1/3} |\sigma|^{-1} \tag{2.17}$$

has not, to our knowledge, been mentioned before.

### 2.3. Relaxation to the stationary state of directed percolation

Relaxation to the stationary state of the directed percolation problem has not been considered before but can be studied by the same techniques. One has to keep the initial time term, with “coupling constant”  $\rho_B^{(0)}$ , in the action of Eq. (1.20). This term represents an initial state with Poissonian distribution of the  $B$  particles. In the diagrams it will be shown as an arrowed leg starting from a  $t = 0$  vertex indicated by an empty circle (see Fig. 1). It is easy to extend the analysis to a non-Poissonian initial distribution of particles. In that case we have to include in the action  $S_{DP}$  of Eq. (1.20) another initial time term, viz.

$$\Delta_\psi \int d^d x \bar{\psi}^2(x, 0). \tag{2.18}$$

Higher-order terms in the field  $\bar{\psi}$  also appear, but are irrelevant under renormalization. The coupling constant  $\Delta_\psi$  in Eq. (2.18) is related to the width of the distribution by

$$\Delta_\psi = -\frac{1}{2\rho} [\langle \Delta n_i^2 \rangle - \langle n_i \rangle] \tag{2.19}$$

which vanishes for a Poisson distribution. The naive scaling dimension of the  $\Delta_\psi$  term is easily found to be zero. This means that it is marginal and has to be included in the RG calculation. Any other terms generated by these two initial terms are irrelevant. Fig. 2 shows the two diagrams that yield renormalization contributions to  $\rho_B^{(0)}$  and to  $\Delta_\psi$  which are *linear* in these quantities. The conclusions of this section may be arrived at on the basis of only these diagrams. Nevertheless, we shall complete the calculation to one-loop order by also computing the contributions of the diagrams of Fig. 3. The diagram of Fig. 3a is the most complicated one to consider and the only one that we shall discuss explicitly. Its expression is

$$2\Delta_\psi \frac{g^2 K_4}{\lambda^2} \int \frac{d^d k}{(2\pi)^d} \int dt d\tau \psi(k, t) \bar{\psi}(-k, t + \tau) \int_{\Omega_1} \frac{d^d q}{(2\pi)^d} e^{-2\lambda q^2 t - \lambda(q^2 - (k-q)^2 + \sigma)\tau}. \tag{2.20}$$

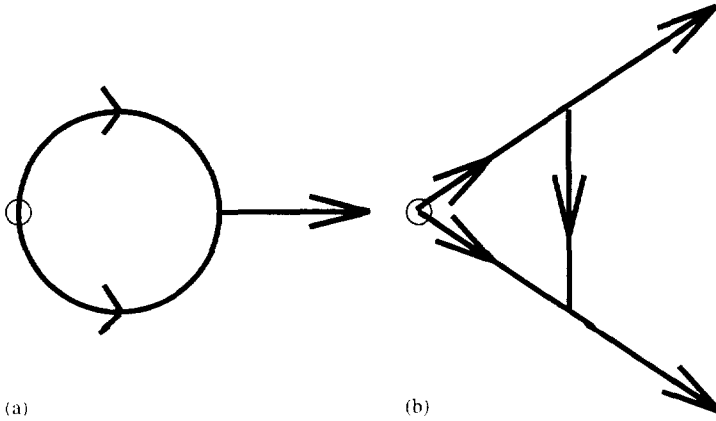


Fig. 2. Diagrams (a) and (b) renormalize  $\rho_B^{(0)}$  and  $A_\psi$ , respectively. They are linear in these quantities.

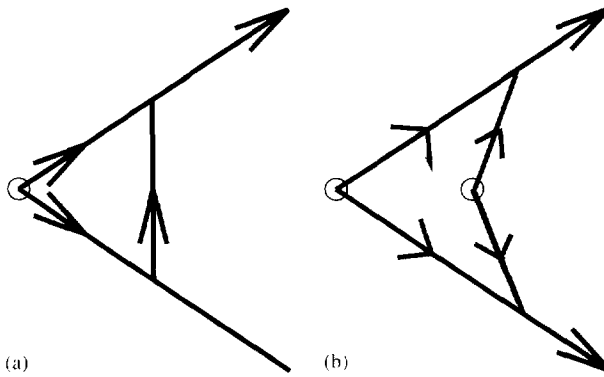


Fig. 3. Other one-loop order diagrams renormalizing  $\rho_B^{(0)}$  (diagram (a)) and  $A_\psi$  (diagrams (a) and (b)).

One expands the fields in the integral for small  $t$  and  $\tau$ , and the exponential in powers of  $k$ . One can then carry the time integrations out explicitly and use the initial time constraint [2]

$$\psi(k, t = 0) = \rho_B^{(0)} - 2A_\psi \bar{\psi}(k, t = 0). \tag{2.21}$$

It appears that out of all terms only

$$2A_\psi \frac{g^2 K_4}{\lambda^2} \rho_B^{(0)} \int d^d x \bar{\psi}(x, 0) \tag{2.22}$$

renormalizes  $\rho_B^{(0)}$  and

$$-4A_\psi^2 \frac{g^2 K_4}{\lambda^2} \int d^d x \bar{\psi}^2(x, 0) \tag{2.23}$$

renormalizes  $\Delta_\psi$ . The other terms are irrelevant. The recursion relations for the couplings  $\rho_B^{(0)}$  and  $\Delta_\psi$  that one finally finds are, to one loop order,

$$\frac{d\rho_B^{(0)}}{d\ell} = \frac{1}{2}(d - \eta)\rho_B^{(0)} + 2\Delta_\psi g \frac{K_4 A^2}{2\lambda(A^2 + \sigma)} - 2\rho_B^{(0)} \Delta_\psi \frac{g^2 K_4}{\lambda^2}, \tag{2.24}$$

$$\frac{d\Delta_\psi}{d\ell} = \Delta_\psi \left( -\eta - 2g^2 \frac{K_4}{\lambda^2} - 4g^2 \Delta_\psi \frac{K_4}{\lambda^2} - \Delta_\psi g^2 \frac{K_4}{\lambda^2} \right). \tag{2.25}$$

The first term in Eq. (2.24) and the term proportional to  $\eta$  in Eq. (2.25) are due to rescaling of the fields and the variables. The remaining terms in Eq. (2.24) come from the diagrams of Figs. 2a and 3a, and those in Eq. (2.25) from the diagrams of Fig. 2b, Fig. 3a, and Fig. 3b, respectively. Upon combining Eq. (2.25) with Eqs. (2.11) and (2.15), one sees that only the fixed point  $\Delta_\psi^* = 0$  is stable, and  $\Delta_\psi$  scales with the negative exponent  $y_{\Delta_\psi} = -\varepsilon/2$  and hence is irrelevant. Substitution of this fixed point value in Eq. (2.24) shows that  $\rho_B^{(0)}$  scales with the exponent

$$y_B = \frac{1}{2}(d - \eta). \tag{2.26}$$

We note in passing that, since the same diagrams renormalize  $g\psi\bar{\psi}^2$  and  $\Delta_\psi\bar{\psi}^2|_{t=0}$ , comparison of Eqs. (2.25) and (2.7) shows that the identity

$$y_{\Delta_\psi} = 2 - z - \frac{\varepsilon}{2} + \frac{\eta}{2} \tag{2.27}$$

must hold to every order in  $\varepsilon$ .

#### 2.4. Initial and long-time scaling of the relaxation

We analyze the relaxation towards the stationary state following Diehl and Ritschel [15] (see also Ref. [16]). To simplify the discussion we place ourselves at the critical density, i.e. put  $\sigma = 0$ . Since  $\rho_B(t)$  is equal to the average of  $\psi(x, t)$  with respect to  $e^{-S_{DP}}$ , it scales with the exponent  $(d + \eta)/2$  and one has the scaling behavior

$$\rho_B(t) = b^{-(d+\eta)/2} F(b^{-\varepsilon} t, b^{(d-\eta)/2} \rho_B^{(0)}) \tag{2.28}$$

valid in the limit  $b \rightarrow \infty$  and for fixed arguments of the function  $F$ . Choosing  $b^\varepsilon = t$  and employing the relation  $\beta = \nu(d + \eta)/2$  leads to

$$\rho_B(t) = t^{-\beta/\nu} F(1, \rho_B^{(0)} t^{\beta/\nu + \theta'}) \tag{2.29}$$

In the limit of large time  $t$ , at fixed initial density  $\rho_B^{(0)}$ , one obtains the usual critical relaxation proportional to  $t^{\beta/\nu}$  at the condition that  $F(x, y) \sim 1$  for  $y \rightarrow \infty$ . In the limit of small  $\rho_B^{(0)}$  at fixed time  $t$  a scaling regime appears whose existence was first pointed out by Janssen et al. [2]. In this limit the density  $\rho_B(t)$  should be proportional to  $\rho_B^{(0)}$ , which is possible only if  $F(x, y) \sim y$  in the small  $y$  limit. Hence, we can write

$$\rho_B(t) = \rho_B^{(0)} t^{\theta'} \mathcal{F}(\rho_B^{(0)} t^{\beta/\nu - \theta'}) \tag{2.30}$$

in which  $\mathcal{F}(y) = y^{-1}F(1, y)$  and

$$\theta' = -\frac{\eta}{z} = \frac{\varepsilon}{12}, \tag{2.31}$$

where the function  $\mathcal{F}$  has the limit behavior

$$\mathcal{F}(y) \sim 1 \quad (y \rightarrow 0), \quad \mathcal{F}(y) \sim y^{-1} \quad (y \rightarrow \infty). \tag{2.32}$$

Here  $\theta'$  is the critical initial slip exponent [2]. The fact that it is positive here implies that in an initial time regime (excluding microscopically small times) the  $B$  particle density *increases* as

$$\rho_B(t) \sim \rho_B^{(0)} t^{\theta'}. \tag{2.33}$$

Crossover to the regime of asymptotically long times takes place at  $t \sim \tau_{\text{cross}}$ , with

$$\tau_{\text{cross}} \sim (\rho_B^{(0)})^{-1/(\beta/\nu z + \theta')}. \tag{2.34}$$

The positivity of  $\theta'$  and the initial rise of  $\rho_B(t)$  have been discussed and interpreted in Refs. [2,15]. They can be qualitatively understood by the lack of critical fluctuations in the initial state. In the case of the reaction–diffusion systems considered here, the fluctuations in the stationary state are responsible for the shift of the critical density to a larger value compared to its mean-field value. This is reflected by the negative  $\sigma^{\xi}$  obtained in Eq. (2.12). Since the density distribution in the initial state is Poissonian and thus free of long-range correlations, the system shows qualitatively a mean-field-like behaviour during the initial stage of the relaxation. For  $\sigma = \sigma^{\xi} < 0$ , the density of  $B$ -particles therefore increases at (macroscopically) short times  $t$ . Notice, however, that this qualitative argument applies generally not to *microscopically* short times [17].

An important point, stressed in Ref. [2] and not apparent from the preceding discussion, is that  $\theta'$  is an *independent* critical exponent. Whereas in the special case of  $S_{DP}$  considered here it could be expressed as  $-\eta/z$  (see Eq. (2.31)), that relation does not hold in general. This will be seen, in particular, in Section 3.

The scaling function  $\mathcal{F}$  is known at the mean-field level,

$$\mathcal{F}(y) = \frac{1}{1 + \text{constant} \times y}. \tag{2.35}$$

In dimension  $d = 4$  one finds that the  $B$  density decays as

$$\begin{aligned} \rho_B(t) &\sim \rho_B^{(0)} \ln^{1/6} t \quad (t \ll \tau_{\text{cross}}), \\ \rho_B(t) &\sim t^{-1} \ln^{1/2} t \quad (t \gg \tau_{\text{cross}}), \end{aligned} \tag{2.36}$$

where  $\tau_{\text{cross}} \sim (\rho_B^{(0)})^{-1} \ln^{1/3}(1/\rho_B^{(0)})$ .

Finally, when the system is slightly off criticality, i.e. for  $\sigma \neq 0$ , the scaling function  $\mathcal{F}$  depends on the supplementary scaling variable  $|\sigma|t^{1/\nu z}$ . The power-law decay  $\sim t^{-\beta/\nu z}$  is then cut off exponentially on a time scale of order  $\xi^z$ , where  $\xi \sim \sigma^{-\nu}$  is the correlation length.

### 3. Wilson renormalization of the full action $S$

#### 3.1. Introduction

The RG procedure applied above to the action  $S_{DP}$  of the directed percolation problem can be extended without difficulty to the action  $S$  given by Eq. (1.16) and describing the full problem defined in Section 1.1. This action contains a parameter  $\mu$  which distinguishes between the diffusion constants of the healthy and the sick individuals. It appears to constitute an immediate and natural step beyond the action of directed percolation: Led by an interpretation different from ours, and starting from a pair of Langevin equations postulated for a problem in population dynamics, Kree et al. [1] arrive at the same action but specialized to  $\mu = 0$ . Using field-theoretic techniques they analyze its stationary state and compute the critical exponents  $z$ ,  $\eta$ , and  $\nu$  to order  $\varepsilon$ .

Below we first derive the stationary-state RG equations for the general case  $\mu \neq 0$ . Since  $\mu$  appears to be relevant, a new analysis is needed. For  $\mu < 0$  we find a new fixed point and calculate the new critical exponents to lowest order in  $\varepsilon$ .

Next we derive the RG equations for the additional couplings that describe the initial state. We shall, in particular, consider an initial particle distribution which is independent from site to site, but non-Poissonian.

#### 3.2. Stationary state of the full action $S$

Our starting point is the action  $S$  of Eq. (1.16). Since our first interest is in the stationary state, we ignore for the moment the initial time term proportional to  $\rho_B^{(0)}$ . As before we write  $S = S_0 + S_{int}$ . The free part  $S_0$  of the action now includes the term in Eq. (1.16) that is proportional to  $\mu$ , and whose presence is due to the diffusion constants of the  $A$  and  $B$  particles being unequal. This term violates the time reversal symmetry of Eq. (1.18) and introduces a nonzero correlator

$$\langle \varphi(k, t_1) \bar{\psi}(-k, t_2) \rangle = \frac{\mu k^2}{(\lambda - 1)k^2 + \lambda \sigma} \Theta(t_1 - t_2) (e^{-k^2(t_1 - t_2)} - e^{-\lambda(k^2 - \sigma)(t_1 - t_2)}). \tag{3.1}$$

Diagrammatically this contraction is represented by a solid leg with an arrow that connects to a dashed leg without arrow to form an internal line (see Fig. 1). As a consequence the symmetry between the  $\psi^2 \bar{\psi}$  and  $\psi \bar{\psi}^2$  vertices in the action is broken and we write the interaction part  $S_{int}$  as

$$S_{int} = \int [g_1 \psi^2 \bar{\psi} - g_2 \psi \bar{\psi}^2 + u \psi \bar{\psi} (\varphi + \bar{\varphi})] \tag{3.2}$$

with independent  $g_1$  and  $g_2$ .

Some further general considerations are useful. The vertices of  $S_{int}$  (see Fig. 1) do not allow for renormalization of the  $\varphi$  propagator. We therefore conclude that

$$z = 2. \tag{3.3}$$

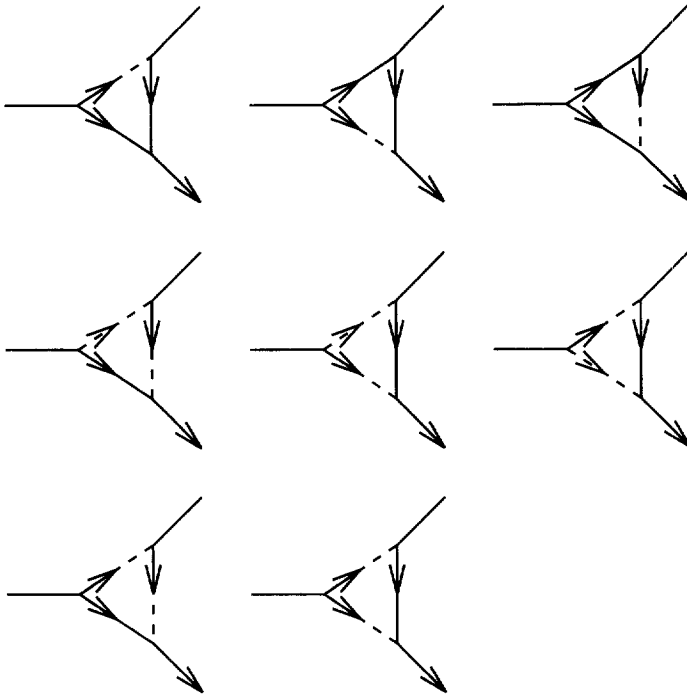


Fig. 4. Additional diagrams entering the renormalization of  $g_1$  for  $\mu \neq 0$  with respect to the  $\mu = 0$  case.

In order to show that this is the dynamic scaling exponent not only of the conserved density  $\varphi$  but also of the fluctuations of the order parameter, one has to check that the coupling  $\lambda$  tends to finite nonzero fixed point value (see below). From dimensional analysis the dimensions of  $\varphi$  and  $\bar{\varphi}$  must be equal to  $d/2$ . However, because of the symmetry breaking discussed above, one has to allow for two distinct anomalous dimensions,  $\eta$  and  $\bar{\eta}$ , of the fields  $\psi$  and  $\bar{\psi}$ .

We have worked out the RG recursion relations for the couplings, taking into account all one loop diagrams (they involve up to three vertices). Some of these diagrams are shown in Fig. 4. For convenience, we put

$$X = \frac{g_1 g_2}{4\lambda^2} K_4, \quad Y = \frac{u^2}{(\lambda + 1)^2} K_4, \quad Z = \frac{g_2 u \mu}{\lambda^2 (\lambda + 1)^2} K_4. \quad (3.4)$$

With this notation we are led to the following recursion relations:

$$\frac{dg_1}{d\ell} = g_1 \left[ \frac{\varepsilon}{2} - \eta - \frac{\bar{\eta}}{2} - 8X + \left( 3 - \frac{1}{\lambda} \right) Y - (5\lambda + 3)Z - \frac{(\lambda + 1)^3 Z^2}{4X} + \frac{(\lambda + 1)^3 YZ}{4\lambda X} \right], \quad (3.5)$$



$$\frac{dg_2}{d\ell} = g_2 \left[ \frac{\varepsilon}{2} - \frac{\eta}{2} - \bar{\eta} - 8X + \left( 3 + \frac{1}{\lambda} \right) Y - 2(2\lambda + 1)Z \right], \tag{3.6}$$

$$\frac{du}{d\ell} = u \left[ \frac{\varepsilon}{2} - \frac{1}{2}(\eta + \bar{\eta}) - 4X + Y - (2\lambda + 1)Z \right], \tag{3.7}$$

$$\frac{d\lambda}{d\ell} = \lambda \left[ -\frac{1}{2}(\eta + \bar{\eta}) - X + \frac{1}{\lambda + 1}Y + \frac{\lambda^2 - 4\lambda - 1}{4(\lambda + 1)}Z \right]. \tag{3.8}$$

Upon requiring that the coefficient of the  $\bar{\psi}\partial_t\psi$  term remains equal to unity we find the supplementary equation

$$0 = -\frac{1}{2}(\eta + \bar{\eta}) - 2X + Y - \frac{1}{2}(3\lambda + 1)Z. \tag{3.9}$$

Finally, the coefficient  $\lambda\sigma$  renormalizes according to

$$\begin{aligned} \frac{d\lambda\sigma}{d\ell} = & \left[ 2 - \frac{1}{2}(\eta + \bar{\eta}) \right] \lambda\sigma + 2\frac{g_1g_2}{2\lambda} \frac{K_4A^4}{A^2 + \sigma} - u^2 \frac{K_4A^4}{(\lambda + 1)A^2 + \lambda\sigma} \\ & + \frac{ug_2\mu}{\lambda} \frac{K_4A^6}{(A^2 + \sigma)[(\lambda + 1)A^2 + \lambda\sigma]}. \end{aligned} \tag{3.10}$$

The preceding equations, together with the appropriate initial conditions for  $\ell = 0$ , determine the RG trajectories. By combining Eq. Eqs. (3.10) and (3.7) we may check, before even solving the full set of equations, that to order  $\varepsilon$

$$y_\sigma = 2 - \frac{\bar{\eta}}{2} - 4X^{\text{fix}} + Y^{\text{fix}} - (2\lambda + 1)Z^{\text{fix}} = 2 - \frac{\varepsilon}{2} \tag{3.11}$$

is independent of  $\mu$ . The existence of a Ward identity (see the appendix in Ref. [1]) which is preserved for  $\mu \neq 0$  guarantees in fact that this result is true to all orders in  $\varepsilon$ .

### 3.2.1. Equal diffusion constants, $\mu = 0$

For  $\mu = 0$  the time-reversal symmetry is restored, we must set  $\eta = \bar{\eta}$ , and the RG Eqs. (3.5) and (3.6) for  $g_1$  and  $g_2$  become identical. The critical behavior is controlled by the fixed point

$$X^{\text{fix}} = \frac{\varepsilon}{4}, \quad Y^{\text{fix}} = \frac{3\varepsilon}{8}, \quad Z^{\text{fix}} = 0, \quad \lambda = 2, \quad \eta = -\frac{\varepsilon}{8} \tag{3.12}$$

first found by Kree et al. [1].

### 3.2.2. Unequal diffusion constants, $\mu \neq 0$

The flow defined by Eqs. (3.5) and (3.8) and subject to condition, Eq. (3.9), possesses a stable fixed point at which  $\lambda$  is the solution of

$$2\lambda^3 - 9\lambda^2 - 6\lambda - 1 = 0, \tag{3.13}$$

i.e.

$$\lambda = [(2 + \sqrt{3})^{1/3} + (2 - \sqrt{3})^{1/3} - 2]^{-1} = 5.10664\dots \tag{3.14}$$

and furthermore

$$X^{\star\star} = \frac{5\lambda^2 + 4\lambda + 1}{8\lambda(3\lambda + 1)}\varepsilon = 0.227\varepsilon, \quad (3.15)$$

$$Y^{\star\star} = \frac{\lambda}{2(\lambda + 1)}\varepsilon = 0.418\varepsilon, \quad (3.16)$$

$$Z^{\star\star} = \frac{\lambda^2 - 2\lambda - 1}{2\lambda(\lambda + 1)(3\lambda + 1)}\varepsilon = 0.0146\varepsilon. \quad (3.17)$$

This leads to the critical exponents

$$\eta = 0, \quad \bar{\eta} = -\frac{\lambda}{3\lambda + 1}\varepsilon = -0.313\varepsilon. \quad (3.18)$$

The result  $\eta = 0$  holds at every order in  $\varepsilon$ . It follows that the exponent  $\beta = \nu(d + \eta)/2$  takes the mean-field value  $\beta = 1$ . In checking the linear stability of this fixed point in the subspace of  $(g_1, g_2, u, \lambda)$ , we found two negative eigenvalues of multiplicity two. It is easy to see that this fixed point governs the critical behavior of only the stationary state with  $\mu < 0$ . Indeed, the unrenormalized ( $\ell = 0$ ) value of  $Z$  has the sign of  $-\mu$ ; and from the definition of  $Z$  combined with Eqs. (3.6) and (3.7) one deduces that  $Z$  cannot change sign under renormalization; so that the fixed point can be reached only if  $Z(\ell = 0)$  is positive just like  $Z^{\star\star}$ .

### 3.3. A first-order transition for $\mu > 0$ ?

We shall now assume that the stationary state of the system undergoes a first order transition for  $\mu > 0$ . This assumption makes of  $\mu = 0$  a tricritical point. We now develop a scaling analysis of the stationary state for  $\mu$  of either sign. In this subsection the exponents describing the  $\mu < 0$  fixed point will bear a minus index.

At the tricritical point  $\mu = 0$  characterized by  $\eta$ ,  $\nu$ , and  $\beta$  the coupling constant  $\mu$  is a relevant perturbation with crossover exponent  $y_\mu = -\eta/2 > 0$ . For both  $\sigma$  and  $\mu$  small compared to microscopic momentum scales  $\rho_B$  therefore has the scaling form

$$\rho_B(\sigma, \mu) = (-\sigma)^\beta \mathcal{F}(\mu(-\sigma)^{-\nu y_\mu}). \quad (3.19)$$

The critical behavior for  $\sigma \rightarrow 0$  at fixed  $\mu < 0$  is described by the exponents  $\eta_-$ ,  $\nu$ , and  $\beta_-$  and we get

$$\rho_B(\infty) \simeq A(-\sigma)^{\beta_-}. \quad (3.20)$$

The amplitude  $A$  is nonuniversal and is a singular function of  $\mu$ , because the limits  $\sigma \rightarrow 0$  and  $\mu \rightarrow 0$  do not commute. If  $\mu$  is small compared to microscopic scales but large compared to  $(-\sigma)^{\nu y_\mu}$  we see from Eq. (3.19) that

$$A \sim (-\mu)^{(\beta_- - \beta_-)/(\nu y_\mu)}. \quad (3.21)$$

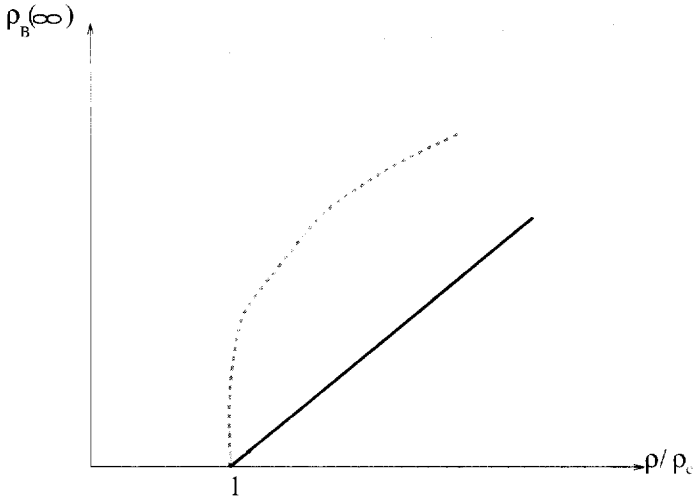


Fig. 5. Stationary-state density  $\rho_B(\infty)$  of the  $B$  particles plotted qualitatively against the total density  $\rho$ . The curve for  $\mu > 0$  (dashed line) rests on the hypothesis of there being a first order transition. The curve for  $\mu = 0$  (dotted line) was obtained in Ref. [1]. The curve for  $\mu < 0$  (plain line) is a mean-field like straight line.

We use analogous arguments in the case  $\mu > 0$  assuming that the phase transition is first order. A first-order transition means that  $\rho_B(\infty)$  has a finite value in the limit  $-\sigma \rightarrow 0$  at fixed  $\mu > 0$ . Eq. (3.19) now gives

$$\rho_B(\infty) \sim \mu^{1/\delta} \tag{3.22}$$

with  $\delta = \nu y_\mu / \beta = -\eta / (d + \eta) > 0$ . Fig. 5 shows a qualitative plot of the resulting behavior of the stationary-state density  $\rho_B(\infty)$  as a function of  $\rho$  in the three regimes  $\mu < 0$ ,  $\mu = 0$ , and  $\mu > 0$ .

### 3.4. Relaxation to the stationary state of the full action $S$

#### 3.4.1. Non-Poissonian initial states: General remarks

We wish to consider an initial particle distribution which, as before, is the product on  $i$  of identical single-site distributions  $p(m_i, n_i)$ , where  $m_i$  and  $n_i$  count the  $A$ 's and  $B$ 's, respectively, on site  $i$ . The distribution  $p$  is arbitrary. It need not be Poissonian in either  $m_i$  or  $n_i$ , and need not factorize. The interest of studying such initial distributions is that all initial distributions with short-range correlations reduce to them under renormalization. The general distribution of this class is represented in the action  $S$  by the initial time terms

$$\int d^d x [-\rho_B^{(0)} \bar{\psi}(x, 0) + \Delta_\phi \bar{\phi}^2(x, 0) + \Delta_{\phi\psi} \bar{\phi}(x, 0) \bar{\psi}(x, 0) + \Delta_\psi \bar{\psi}^2(x, 0)] \tag{3.23}$$

plus terms of higher order in  $\bar{\varphi}$  and  $\bar{\psi}$ , which are irrelevant under renormalization. Here, setting  $\ell_i = m_i + n_i$ , we have

$$\begin{aligned} \Delta_\psi &= -\frac{1}{2\rho}[\langle \Delta n_i^2 \rangle - \langle n_i \rangle], \\ \Delta_\varphi &= -\frac{1}{2\rho}[\langle \Delta \ell_i^2 \rangle - \langle \ell_i \rangle], \\ \Delta_{\varphi\psi} &= -\frac{1}{\rho}[\langle \Delta n_i^2 \rangle - \langle n_i \rangle + \langle \Delta m_i \Delta n_i \rangle]. \end{aligned} \tag{3.24}$$

If the  $A$  and  $B$  particles have uncorrelated Poisson distributions, then  $\Delta_\varphi = \Delta_{\varphi\psi} = \Delta_\psi = 0$ .

### 3.4.2. Non-Poissonian initial $A$ distribution

For the sake of simplicity, we focus on the  $\mu = 0$  case. As discussed above, in an arbitrary initial state we have to deal with the four parameters  $\rho_B^{(0)}$ ,  $\Delta_\varphi$ ,  $\Delta_{\varphi\psi}$ , and  $\Delta_\psi$ . Also for simplicity, and because it is one of the interesting cases, we shall consider here  $\Delta_\psi = \Delta_{\varphi\psi} = 0$ . Eq. (3.24) implies that this corresponds to having a Poisson distribution of the  $B$  particles, a non-Poissonian distribution of the  $A$  particles, and no correlations between the occupation numbers of the two species. It then follows that

$$\Delta_\varphi = -\frac{1}{2\rho}[\langle \Delta m_i^2 \rangle - \langle m_i \rangle]. \tag{3.25}$$

The  $\Delta_\varphi$  term allows for contractions of the  $\varphi$  between themselves according to

$$\langle \varphi(k_1, t_1) \varphi(k_2, t_2) \rangle = -\Delta_\varphi (2\pi)^d \delta^{(d)}(k_1 + k_2) \Theta(t_1) \Theta(t_2) e^{-k_1^2(t_1 + t_2)}. \tag{3.26}$$

There are no diagrams by which the  $\Delta_\varphi$  term gets itself normalized: it is strictly marginal. However, the new contractions of Eq. (3.26) contribute to the renormalization of the  $\rho_B^{(0)}$  term. The additional diagrams that appear are shown in Fig. 6 The explicit expression of diagram Fig. 6b reads

$$u^2 \Delta_\varphi \int \frac{d^d k}{(2\pi)^d} \int dt d\tau \psi(k, t) \bar{\psi}(-k, t + \tau) \Theta(t) \Theta(\tau) \int_{q \in \Omega_1} e^{-2q^2 t - \lambda[(q-k)^2 + \sigma]\tau - q^2 \tau}. \tag{3.27}$$

One repeats the analysis performed in Section 2.3; however, unlike diagram Fig. 4a that appears in the directed percolation case, diagram Fig. 6b yields a renormalization of  $\rho_B^{(0)}$  that does not vanish at the fixed point. The recursion relation for the  $\rho_B^{(0)}$  vertex is

$$\frac{d\rho_B^{(0)}}{d\ell} = \frac{d - \eta - \eta_0}{2} \rho_B^{(0)}, \tag{3.28}$$

where we have set

$$\eta_0 \equiv (\lambda + 1) Y^{\hat{\nu}} \Delta_\varphi = \frac{9}{8} \Delta_\varphi \varepsilon. \tag{3.29}$$

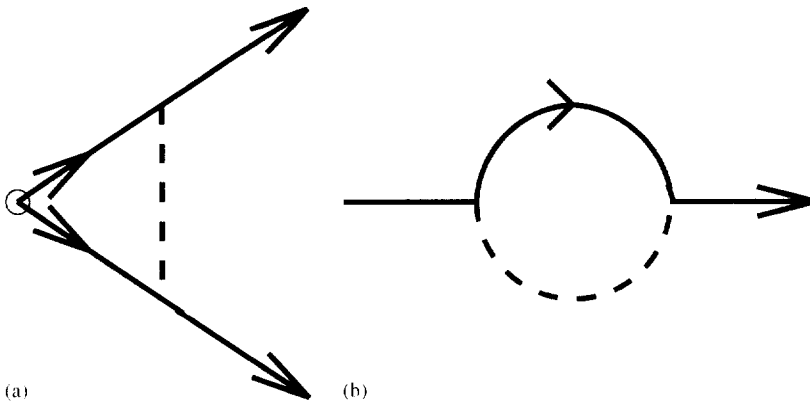


Fig. 6. Diagram (a) renormalizes  $A_\psi$ . Diagram (b) renormalizes  $\rho_B^{(0)}$  and  $A_\psi$ .

From Eqs. (3.28) and (3.29) we conclude that the scaling dimension of  $\rho_B^{(0)}$  is

$$y_B = \frac{d}{2} - \frac{\eta}{2} - \frac{\eta_0}{2}. \tag{3.30}$$

This exponent is therefore nonuniversal. We may repeat the scaling argument of Section 2.4, which allows us to conclude that the critical initial slip exponent is

$$\begin{aligned} \theta' &= -\frac{d + \eta}{2} + y_B = -\frac{\eta}{z} - \frac{\eta_0}{2z} \\ &= -\frac{2 - 9A_\varphi \varepsilon}{32}, \end{aligned} \tag{3.31}$$

where in the last equality we have used that  $z = 2$ . We have therefore exhibited a nonuniversal exponent that characterizes the short-time behavior of the density of the  $B$  species. From expression (Eq. (3.25)), combined with the fact that at criticality the density of  $A$  particles differs negligibly from  $\rho$ , one may deduce that

$$A_\varphi \leq \frac{1}{2}\rho \tag{3.32}$$

so that  $\theta'$  has the (nonuniversal) lower bound  $\theta'_{\min} = (4 - 9\rho)\varepsilon/64$ . The coupling  $A_\varphi$  attains the maximum value allowed by Eq. (3.32), and hence  $\theta'$  its minimum, when the  $m_i$  have the probability law

$$p_A(m_i) = (1 - \rho)\delta_{m_i,0} + \rho\delta_{m_i,1} \tag{3.33}$$

(where we take  $\rho < 1$ ). This means that the  $A$  particles are randomly distributed but without any multiple occupancy. Apparently, this is the environment giving the slowest initial rise of the number of infected individuals.

### 3.4.3. Relaxation for $\mu < 0$

We shall now consider critical relaxation to the stationary state for  $\mu < 0$ , but limit ourselves to Poissonian initial distributions:  $A_\psi = A_\varphi = A_{\varphi\psi} = 0$ . The initial  $B$  particle density renormalizes according to

$$\frac{d\rho_B^{(0)}}{d\ell} = \frac{1}{2}(d - \bar{\eta})\rho_B^{(0)}. \quad (3.34)$$

After slightly generalizing the reasoning of Section 2.4, we conclude that there is a nontrivial critical initial slip exponent

$$\theta' = -\frac{\bar{\eta}}{2z} = \frac{\lambda}{4(3\lambda + 1)}\varepsilon = 0.0782\varepsilon \quad (\mu < 0). \quad (3.35)$$

To summarize, in the initial stage for  $\mu < 0$ ,  $\rho_B(t) \sim \rho_B^{(0)}t^{\theta'}$ , then, over a time scale  $\xi^z$ , the density decays as  $\rho_B(t) \sim t^{-d/4}$ . This last result holds at every order in  $\varepsilon$ .

## 4. Conclusion

We have adapted Wilson's renormalization scheme to the two-species reaction–diffusion process  $A + B \rightarrow 2B$ ,  $B \rightarrow A$ . The stationary state of this process exhibits a second-order phase transition at which the  $B$ s become extinct. A parameter  $\mu = 1 - D_B/D_A$  controls the relative strength of the diffusion constants  $D_A$  and  $D_B$  of the two species. We have determined the critical behavior of the stationary state for  $\mu < 0$ , that is, when the species subject to extinction ( $B$ ) diffuses faster than the species ( $A$ ) which it parasites on. Special emphasis has been put on the relaxation from various initial states to the critical stationary state. As it happens in magnetic systems, we found an initial relaxation regime which scales with an independent exponent  $\theta'$ , arising from renormalization of the initial time terms in the action. The reaction–diffusion process has an order parameter (the  $B$  particle density) that couples to a conserved quantity (the total density). We have shown that  $\theta'$  varies continuously with the width of the distribution of the conserved density. The same effect occurs in magnetic systems with a continuous symmetry of the order parameter, where the symmetry implies the existence of a conserved quantity coupling to the order parameter [18].

We found that for positive  $\mu$  the renormalization transformation has no stable fixed point, which signals the absence of a second-order phase transition. We therefore discussed the possibility of a first-order transition; this scenario is plausible but we lack decisive arguments in its favor. This unresolved point deserves further attention.

A question of great interest concerns the low-dimensional behavior of this reaction–diffusion system. The  $\varepsilon$  expansion is meaningful slightly below the upper critical dimension  $d_c = 4$ , and one may imagine that it still gives a fairly good idea of what happens in  $d = 3$ . However, the qualitative picture provided by the  $\varepsilon$  expansion may break down below some new critical dimension  $d'_c$ . Eq. (1.19) shows that the (unrenormalized) action contains terms, neglected in this work, that become relevant with respect to the

free (Gaussian) theory when  $d=2$ . It is therefore possible that low dimensions require separate consideration [19]. Even the seemingly simple one-dimensional version of this model exhibits nontrivial behavior [20] that remains to be elucidated.

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