Universality Class of Nonequilibrium Phase Transitions with Infinitely Many Absorbing States

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We consider systems whose steady states exhibit a nonequilibrium phase transition from an active state to one—among an infinite number—absorbing state, as some control parameter is varied across a threshold value. The pair contact process, stochastic fixed-energy sandpiles, activated random walks, and many other cellular automata or reaction-diffusion processes are covered by our analysis. We argue that the upper-critical dimension below which anomalous fluctuation driven scaling appears is $d_c = 6$, in contrast to a widespread belief. We provide the exponents governing the critical behavior close to or at the transition point to first order in an $\varepsilon = 6 - d$ expansion.

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Absorbing-state phase transitions.—Nonequilibrium phase transitions between an active and an absorbing state are encountered in a variety of fields ranging from chemical kinetics to the spreading of computer viruses [1]. From a theoretical standpoint absorbing-state transitions form natural counterparts to equilibrium phase transitions. The transition rates used in the stochastic dynamics employed to model the physical phenomenon under consideration do not satisfy detailed balance (with respect to an *a priori* defined energy function). In spite of this apparent freedom, the number of universality classes that the transition can fall into is incredibly small. Among known universality classes, that of directed percolation (DP) is by far the broadest. And indeed, in the absence of additional symmetries or conservation laws, as was conjectured 20 years ago by Grassberger [2], an absorbing-state transition will invariably fall into the DP universality class. Only if the dynamics possess additional features such as particle number parity conservation, or the coupling to an auxiliary field (whether conserved or not, static or diffusing, etc.), or the existence of an infinite number of absorbing states, will the transition belong to a new class. Apart from the formal interest in classifying nonequilibrium phase transitions and in identifying which microscopic ingredients make a transition belong to a given universality class, there exists a greater challenge. As recently summarized by Hinrichsen [3], in spite of its domination in the theoretical physics literature, the directed percolation universality class was actually never observed in a real experiment. This is often attributed to the presence of defects in a real experiment (DP is known to be very sensitive to quenched disorder) or to other ill-controlled effects, such as a hidden conservation law or a coupling to an auxiliary field.

On one hand, the interest in absorbing-state transitions was recently enhanced by the discovery by Vespignani and co-workers [4] of their relationship with the ubiquitous phenomenon of self-organized criticality (SOC) [5]. It was established that the scaling behavior observed there was entirely governed by an underlying phase transition (thereby, incidentally, tempering the mystics of SOC). On the other hand, in a separate wave of articles, research has focused on absorbing-state transitions in which the order parameter freezes into one among an infinity of absorbing states, but without any additional conservation law. The paradigmatic example of a system showing such a behavior is the pair contact process, initially introduced by Jensen and Dickman [6], for which Muñoz and coworkers [7] devised a convincing phenomenological picture that we shall later use as our starting point.

The existence of an infinite number of absorbing states (in the large-system limit) and the coupling to an auxiliary static field are the common characteristics to the microscopic models that we wish to investigate here.

It is the purpose of the present work to provide a full renormalization group picture of the phase transition at work in systems possessing an infinite number of absorbing states, with or without an additional conservation law. We shall rely on a phenomenological Langevin approach as a starting point. From there we shall show how renormalization group arguments can be applied. This will lead us to finding the upper-critical dimension of those models. Then we will sketch the reasoning leading to the expression of the critical exponents within the framework of an expansion around the upper-critical dimension. As will be clear there is interesting new physics to learn from the many technical obstacles that pave the way to the full scaling picture.

We now turn to a presentation of two microscopic models chosen for their representativeness and ease of formulation in which the absorbing-state phase transitions we wish to study appear. We also introduce the Langevin equations encoding their dynamics. Then we sketch the field-theoretic line of reasoning leading to the computation of the critical exponents. Finally, we provide a critical discussion of our results in the light of the existing literature.

Two models and their phenomenological coarse-grained description.—In the pair contact process (PCP), particles are thrown on a lattice. A pair may either

annihilate $A + A \to \varnothing$ or produce a single offspring $A + A \to A + A + A$. Each lattice site can be occupied by at most one particle. Since isolated particles cannot diffuse, a configuration in which pairs of nearest neighbors are absent remains frozen in time. The order parameter ψ is local density of pairs of nearest neighbor particles and the control parameter is the branching over annihilation rate ratio. The Langevin equation believed to describe the dynamics of ψ was coined by Muñoz *et al* [7]. It reads

$$\text{PCP: } \partial_t \psi(\mathbf{r}, t) = D\Delta_{\mathbf{r}}\psi(\mathbf{r}, t) - \sigma\psi(\mathbf{r}, t) - g_1\psi^2(\mathbf{r}, t) - g_3\psi(\mathbf{r}, t)e^{-\int_0^t dt'\psi(\mathbf{r},t')} + \eta(\mathbf{r}, t), \quad (1)$$

with η a Gaussian white noise whose correlations are $\langle\eta(\mathbf{r}, t)\eta(\mathbf{r}', t')\rangle = g_2\psi\delta^{(d)}(\mathbf{r} - \mathbf{r}')\delta(t - t')$. The coefficients σ, g_1, g_2, and g_3 are coarse-grained analogs of the reaction rates. The memory term is the signature of the feedback of isolated particles on the pair dynamics. For a detailed explanation of the physical origin of the various contributions appearing in Eq. (1) we refer the reader to [7]. The list of microscopic models whose coarse-grained description is believed to be encoded in the Langevin Eq. (1) also includes the transfer threshold process [8] and various models for catalysis (the dimer reaction [6], the dimer-dimer [9], or the dimer-trimer models [10]). A detailed analysis of the mean-field properties of this equation was provided by Muñoz *et al.* [7].

The second family of models that we are interested in is embodied by the so-called Manna or stochastic fixed-energy sandpiles (FES) [4,11,12]. Grains of energy are initially randomly distributed on a lattice. Whenever a lattice site is occupied by more than two particles (in dimension d) the excess particles randomly hop to a nearest neighbor site. The number of active sites plays the role of the order parameter. The total number of particles is strictly conserved. An evolution equation for the local density of active sites ψ was recently proposed by Dickman *et al.* [4]. It reads

$$\text{FES: } \partial_t \psi = D\Delta_{\mathbf{r}}\psi(\mathbf{r}, t) - \sigma\psi(\mathbf{r}, t) - g_1\psi^2(\mathbf{r}, t) + g_4\psi(\mathbf{r}, t)\int_0^t dt'\,\Delta_{\mathbf{r}}\psi(\mathbf{r}, t') + \eta(\mathbf{r}, t), \quad (2)$$

with η's correlations having the same expression as in the PCP case and the coefficients σ, g_1, g_2, g_4 are coarse-grained analogs of the microscopic transition rates which depend on the conserved quantity. The nonlocal memory term expresses that space fluctuations of the static and conserved field have a feedback effect upon the order parameter dynamics. Again a rich variety of models were shown to be described by the same coarse-grained dynamics, such as the conserved transfer threshold process, activated random walkers [13], or some model for epidemic spreading in which healthy individuals are static [14].

There are some hazards in relying on naïvely built phenomenological equations as many other interactions are generated by a coarse-graining procedure. In principle, all terms allowed by symmetries should be included. We shall later see that, indeed, some relevant symmetry-preserving terms have to be considered. Let us now recall the common mean-field behavior of those models. Denoting for both models by σ the deviation of the control parameter from its critical threshold value, in the steady state the order parameter behaves as

$$\psi\begin{cases} \propto |\sigma|^{\beta}, & \sigma \to 0^-, \\ = 0, & \sigma \geq 0. \end{cases} \quad (3)$$

In mean field we simply have $\psi = -\sigma/g_1$. At the critical point, the order parameter decays according to

$$\psi(t) \sim t^{-\delta}, \quad t \text{ large}. \quad (4)$$

The correlation length and the relaxation time diverge as $\sigma \to 0$ according to $\xi \sim |\sigma|^{-\nu}$ and $\tau \sim |\sigma|^{-\nu z}$, respectively. Within the framework of a mean-field analysis one finds the following values for the critical exponents: $\beta = 1$, $\delta = 1$, $\nu = \frac{1}{2}$, and $z = 2$.

Analytic strategy.—The interplay between short-time and short-distance fluctuations with long range correlations lies at the heart of the anomalous scaling observed in the vicinity of a critical point ("anomalous" to be understood as non-mean-field). Short-distance singularities (usually referred to as ultraviolet divergencies) govern the way scaling properties deviate from their mean-field expressions. The usual strategy is to retain the leading UV divergencies and to perform a renormalization group analysis with those only. In the vast majority of cases, this is sufficient to reach physical conclusions. Sometimes, however, contributions that were thrown away in the course of the analysis are crucial in preserving the correct physics, though they are irrelevant in determining the renormalization group fixed point. This is the present situation.

Let us start with the PCP. The picture is the following. We expand the exponential memory term and find that the contribution $g_3\psi\int_0^t \psi$ exhibits the leading short-time and short-distance singularity. Power counting shows that the bare dimension of $g_2 g_3$ is $d - 6$ (in units of a length). This type of interaction is known to describe the dynamical percolation universality class [15,16], which has an upper-critical dimension $d_c = 6$. Yet, in the present case, the phase diagram would not be reproduced correctly if only the truncated expansion of the exponential were kept. As seen on the mean-field evolution equation, and as already noted many times in the literature [7], the asymptotics are governed by the local nonlinear term $-g_1\psi^2$ in Eq. (1). However, $g_1 g_2$ has bare dimension $d - 4$ which signals that it is in fact a dangerously irrelevant coupling, along with the subsequent powers of the argument of the exponential. The coupling g_1 is not relevant in determining the underlying fixed-point structure but essential in preserving the overall phase diagram.

An important consequence is that the usual scaling assumption for the order parameter

$$\langle \psi(t) \rangle = b^{-(d+\eta)/2} \mathcal{F}(b^{-z}t, b^{1/\nu}|\sigma|), \quad b \text{ large} \qquad (5)$$

must be abandoned since the scaling function $\mathcal{F}$ will exhibit a singular behavior in the g_1 variable as the latter approaches 0. In mean field $\mathcal{F}$ depends on g_1 as $1/g_1$. While irrelevant variables are traditionally omitted from the list of arguments of $\mathcal{F}$, in the present case this would lead to unphysical conclusions. The next step is to see how this dangerously irrelevant coupling g_1 is renormalized to 0 in the vicinity of the dynamical percolation fixed point (a necessity overlooked in [7]). In order to do this we have followed the technical procedure recalled by Janssen and Schmittmann [17] (see also the references therein). We skip all technical steps [18], and merely quote the final result: $g_1(b) \sim b^{-2-(\varepsilon/7)}$ as the coarse-graining scale b is increased. We have denoted by $\varepsilon = 6 - d$ and the result for the exponent is given to first order in ε. In order to obtain the leading ε behavior we have combined the mean-field expression for the scaling function $\mathcal{F}$ appearing in Eq. (5) as far as its $g_1(b)$ dependence is concerned with the scaling properties of the field, time, and σ at the dynamical percolation fixed point. This has led us to the following critical exponents:

$$\beta = 1 - \frac{3}{14}\varepsilon, \quad \delta = 1 - \frac{1}{4}\varepsilon, \qquad (6)$$

the expressions of which are given to leading order in $\varepsilon = 6 - d$.

As far as the FES described by Eq. (2) are concerned, the situation is a bit more delicate. A rather involved RG analysis [18] shows that in fact the dynamical percolation vertices are generated already at one-loop order (the one-loop graph obtained by connecting g_2 and two g_4's leads to effective g_1 and g_3 vertices, and no effective g_4). This is because, as is very often the case with gradient interactions, the short-scale behavior of the memory term $-g_4\psi(\mathbf{r}, t)\int_0^t dt' \Delta_{\mathbf{r}}\psi(\mathbf{r}, t')$ converts into an effective $-g_3\psi(\mathbf{r}, t)\int_0^t dt' \psi(\mathbf{r}, t')$ contribution (and higher powers) after coarse graining. This also explains the failure of power counting directly on (2) which, naively performed, would lead to the erroneous conclusion that $d_c = 4$ since $g_2 g_4$ has bare dimension $d - 4$ while $g_2 g_3$ has dimension $d - 6$. And then we may apply the reasoning of the previous paragraph and the results of Eq. (6) continue to hold for FES.

What the field-theory approach teaches us can be summarized as follows. First, a detailed analysis of the renormalized interactions shows that both classes of models considered are described by the same-field theory which has an upper-critical dimension $d_c = 6$. This result is in contradiction with the existing literature of the last ten years [19]. Critical exponents are expected to behave differently from mean field in space dimensions $d < 6$. We have determined the critical exponents of the phase transition to leading order in $\varepsilon = 6 - d$. As to FES-like systems, at the renormalization flow fixed point, the conservation law is irrelevant. This seemingly innocuous property leads to technical difficulties since the conservation law is crucial in saving the overall phase diagram. In both the PCP and the FES cases, while the underlying fixed point is that of dynamical percolation, computing the critical exponents requires one to perform an independent renormalization of an irrelevant coupling. Thus the critical exponents cannot be deduced from those of percolation by means of hyperscaling relations. The validity of our approach holds in the vicinity of $d_c = 6$. As d is lowered, already at $d = 4$ new qualitative features show up in the theory. Since the percolation fixed point becomes trivial in $d = 1$ an entirely new picture must inevitably set in as dimension is lowered, perhaps as early as d is decreased below $d = 4$, and possibly again at $d = 2$. Splitting of the universality classes of the PCP and FES in low space dimensions must be envisaged as well.

There are several criticisms that can be opposed to the present work. First of all, the use of phenomenological Langevin equations sometimes proves hazardous, but, fortunately other formalisms [20] allow exact mappings onto field theories having exactly the same features as the ones discussed in this work. Indeed, the effective Langevin Eq. (2) used for sandpiles is, strictly speaking, not correct, since coarse graining (loop corrections) shows that other dominant terms are generated. Most importantly, however, we must admit that there is absolutely no numerical evidence supporting our findings. An obviously too concise summary of the numerical state of affairs is that both FES [21] and PCP belong to the DP class in dimension one (recent studies disagree with the results of [21], such as [22]). But as dimension is increased PCP is still found to belong to the DP class while FES are found to form an independent universality class. Lübeck, in some recent simulations [23], claims that $d_c = 4$ by performing simulations directly in high dimensions. A cheap way out for the theoretician is to refer the reader to a recent paper by Grassberger [24] in the somewhat different but related context of forest fires in which dynamical percolation plays some role as well. There numerical proof is provided that it might well be impossible with present day computers to ever reach the true asymptotic scaling regime. We recall that there are many obstacles on the numerical side: the impossibility to use simple finite-size scaling relations (due to the presence of a dangerously irrelevant operator). Quenched disorder (for FES [25]) and other long-term memories are known to be difficult to overcome numerically, and finally the proximity of the directed percolation fixed point whose influence must be felt until the system eventually crosses over to its actual asymptotics. All of those features, we fear, play a part in rendering the reaching of

the true asymptotics a hopeless endeavor. As far as the observed lack of universality of spreading exponents is concerned [7] we believe that the phenomenon can be understood within a renormalization group picture. As demonstrated in a much simpler case in [27] spreading exponents exhibit nonuniversal values that depend on the moments of the initial distribution of both the order parameter and auxiliary fields. This dependence is all the stronger as the initial distribution deviates from a strict Poissonian law. In less favorable cases the initial distribution introduces couplings at the initial time that cannot even be renormalized, thus questioning the existence of a scaling regime.

As a conclusion we summarize our findings. We have provided field-theoretic arguments showing that a large number of stochastic processes exhibiting a phase transition between an active state and an absorbing state (picked up among an infinite number of such) belong to the universality class of dynamical percolation and therefore have $d_c = 6$ as their upper-critical dimension (instead of $d_c = 4$ as appears in the last decade literature). The existence of a static field to which the order parameter is coupled is the common feature to all considered models. We have provided expressions for the critical exponents within the framework of renormalized perturbation theory in the vicinity of the upper-critical dimension to leading order in $\varepsilon = 6 - d$. A technical account of the results presented in this Letter is in preparation [18].

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