Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Energy flux distribution in a two-temperature Ising model

Vivien Lecomte[1], **Zoltán Rácz**[2] **and Frédéric van Wijland**[1,3]

[1] Laboratoire de Physique Théorique, CNRS UMR 8627, Université de Paris-Sud, 91405 Orsay cedex, France
[2] Institute for Theoretical Physics, HAS Research Group, Eötvös University, 1117 Budapest, Pázmány sétány 1/a, Hungary
[3] Pôle Matière et Systèmes Complexes, CNRS UMR 7057, Université Denis Diderot (VII), 4 place Jussieu, 75005 Paris, France
E-mail: lecomtev@th.u-psud.fr, racz@general.elte.hu and fvw@th.u-psud.fr

Received 20 December 2004
Accepted 15 February 2005
Published 23 February 2005

Online at stacks.iop.org/JSTAT/2005/P02008
doi:10.1088/1742-5468/2005/02/P02008

Abstract. The nonequilibrium steady state of an infinite range Ising model is studied. The steady state is obtained by dividing the spins into two groups and attaching them to two heat baths generating spin flips at different temperatures. In the thermodynamic limit, the resulting dynamics can be solved exactly, and the probability flow in the phase space can be visualized. We can calculate the steady state fluctuations far from equilibrium and, in particular, we find the exact probability distribution of the energy current in both the high and low temperature phases.

Keywords: transport processes/heat transfer (theory), stationary states (theory)

J. Stat. Mech. (2005) P02008

 1742-5468/05/P02008+16$30.00

Contents

1. Introduction

The infinite range Ising model, in which each individual spin interacts with the remaining $N-1$ ones, has served as a useful testbench for many ideas in various subfields of statistical physics, ranging from critical dynamics to spin glasses. The reason is twofold: it is relatively easy to come up with exact yet nontrivial results for this system (in the large system limit at least), while at the same time it represents a genuine interacting system which further possesses a phase transition from a disordered into an ordered state as the temperature is lowered. In the present work we will follow Ruijgrok and Tjon [1] who, by endowing the system with spin-flip dynamics, provided the first example of an exactly solvable critical dynamics problem back in 1973. In the meantime the subjects of interest have drifted towards other issues, but the technical motivations for using the infinite range Ising model remain. Our physical motivations have their roots in the study and identification of the generic properties of nonequilibrium steady states (NESS).

Among the very few exact statements that can be made on NESS, the recently developed fluctuation—or Gallavotti–Cohen—theorem [2] plays a prominent role. This theorem involves a symmetry property of the entropy current distribution function. As a result it provides, at least formally, a prescription for obtaining an infinite set of Green–Kubo-like relations connecting current fluctuations to the system's response upon external forcing. While establishing a Gallavotti–Cohen theorem for Markov processes can be achieved in general terms [3, 4], explicit computations of current distribution functions are rare. For systems *close* to equilibrium such as boundary driven lattice gases, Bodineau and Derrida [5], and Bertini *et al* [6] have provided a general method for computing current distribution functions from the sole knowledge of the Onsager response coefficients. However for the generic case of systems maintained in a steady state *far* from equilibrium, no general principle has hitherto been unravelled. To the best of our knowledge a single exact calculation exists for the totally asymmetric exclusion process, a degenerate case for which the Gallavotti–Cohen theorem (in the version of Lebowitz and Spohn [4]) does not hold. For this reason we have sought to exhibit a system driven far from equilibrium for which such a current distribution function—albeit mean field—is accessible: an Ising model in which spins are connected to heat baths at different temperatures.

There are several ways of driving the infinite range Ising model into a NESS. One that is inspired from a series of recent works [7]–[10] consists in coupling spins to independent heat baths, thus creating a macroscopic energy current by means of a bulk drive. In the particular version we have developed, N Ising spins σ_i have a ferromagnetic interaction energy

$$\mathcal{H}[\{\sigma_i\}] = -\frac{1}{2N}\left(\sum_i \sigma_i\right)^2 \tag{1}$$

and the dynamics of $N/2$ spins is generated by a heat bath of inverse temperature β_1 while the dynamics of the remaining half of the spins is driven by another heat bath at a different inverse temperature β_2. This state of affairs, namely the existence of two heat baths at unequal temperatures, leads to a steady energy current flowing through the spins from the warmer bath towards the colder one.

The physical results that we have obtained are concerned with the steady state measure and the energy current distribution function. We have been able to provide an exact solution to the Fokker–Planck equation governing the probability distribution of magnetization fluctuations, thus leading to the first example of an N-body nonequilibrium system for which probability flow lines in phase space can actually be visualized. Our second achievement is to have provided the large deviation function of the energy current. The methods we have resorted to rest on the various formulations of the master equation governing the microscopic dynamics. On one hand, by means of a Van Kampen [11] expansion of the magnetizations around their mean values, we have obtained a solvable Fokker–Planck equation describing steady state fluctuations. Technically, this amounts to finding an eigenfunction of the master equation evolution operator. On the other hand, by means of a mapping of the master equation onto a quantum Hamiltonian, we have determined the energy current distribution function, which, in technical terms, has proved to be an eigenvalue problem.

Our first task will be to provide an accurate description of the stationary state distribution (section 3), which we will present after having precisely characterized our model and its steady state properties (section 2). Section 4 will be devoted to a complete study of the energy current distribution function, in the light of the Gallavotti–Cohen theorem. We then conclude by listing interesting prospects.

J. Stat. Mech. (2005) P02008

2. A two-temperature Ising model: phase diagram and steady state properties

2.1. Microscopic dynamics

The energy of a configuration of N Ising spins $\boldsymbol{\sigma} = \{\sigma_i\}$ is given by

$$\mathcal{H}[\boldsymbol{\sigma}] = -\frac{M^2}{2N}, \qquad M = \sum_{i=1}^{N} \sigma_i. \tag{2}$$

We now divide the N spins into two groups with labels 1 and 2 of $N/2$ spins each. A spin σ_j from set 1 flips with a rate

$$\forall j \in 1, \qquad w_1(\sigma_j \to -\sigma_j) = \mathrm{e}^{-\beta_1 \sigma_j M/N}. \tag{3}$$

Spins from group 1 try to equilibrate at inverse temperature β_1 with respect to $\mathcal{H}$. Similarly, a spin from group 2 flips according to

$$\forall j \in 2, \qquad w_2(\sigma_j \to -\sigma_j) = \mathrm{e}^{-\beta_2 \sigma_j M/N}. \tag{4}$$

This is the infinite range counterpart to the one-dimensional systems considered by Rácz and Zia [7], and Schmittmann and Schmuser [8]. Writing

$$\beta = \frac{\beta_1 + \beta_2}{2}, \qquad \varepsilon = \frac{\beta_1 - \beta_2}{2} \tag{5}$$

we see that when the temperatures are equal, $\beta_1 = \beta_2 = \beta$, or $\varepsilon = 0$, the system reaches equilibrium at temperature β. This is because the rates (3), (4) then satisfy detailed balance with respect to the Gibbs distribution $Z^{-1}\mathrm{e}^{-\beta\mathcal{H}}$. Though the precise expressions of the rates we have chosen differ from the original Glauber rates, they possess the same qualitative properties with some advantages in the large system limit discovered by Ruijgrok and Tjon [1].

2.2. Phase diagram

Introducing the mean magnetizations

$$m_1 = \frac{1}{N}\left\langle \sum_{j\in 1} \sigma_j \right\rangle, \qquad m_2 = \frac{1}{N}\left\langle \sum_{j\in 2} \sigma_j \right\rangle, \qquad m = m_1 + m_2 \tag{6}$$

we may find the following evolution equations for the averages:

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} = -2m_1 \cosh \beta_1 m + \sinh \beta_1 m, \qquad \frac{\mathrm{d}m_2}{\mathrm{d}t} = -2m_2 \cosh \beta_2 m + \sinh \beta_2 m \tag{7}$$

from which one deduces that in the steady state (provided it exists)

$$m = \tfrac{1}{2}(\tanh \beta_1 m + \tanh \beta_2 m). \tag{8}$$

Interestingly, although the transition rates (3), (4) are different from the standard Glauber rates, they lead to the same steady state average magnetization. From (8) we deduce that in the steady state the system undergoes a second-order phase transition from a high temperature disordered state at $\beta < 1$ in which $m_1 = m_2 = m = 0$ to a low temperature

J. Stat. Mech. (2005) P02008

ordered (doubly degenerate) state at $\beta > 1$ with nonzero magnetizations. In the $\beta \to 1^+$ limit at ε fixed one finds

$$m \simeq \pm\frac{\sqrt{3}}{\sqrt{1+3\varepsilon^2}}\sqrt{\beta-1}, \qquad m_1 \simeq \pm\frac{1+\varepsilon}{2}\frac{\sqrt{3}}{\sqrt{1+3\varepsilon^2}}\sqrt{\beta-1},$$
$$m_2 \simeq \pm\frac{1-\varepsilon}{2}\frac{\sqrt{3}}{\sqrt{1+3\varepsilon^2}}\sqrt{\beta-1}. \tag{9}$$

According to the magnitude of the nonequilibrium drive ε it may be seen that the ordered state may be either ferromagnetic ($|\varepsilon| < 1$) or antiferromagnetic ($|\varepsilon| > 1$, if one allows for negative temperatures, as we shall discuss in our conclusion in section 5).

2.3. Entropy and energy currents

Following the prescription of Lebowitz and Spohn [4] we may define a time integrated instantaneous entropy current by

$$\mathcal{Q}_S(t) = \ln \frac{W(\boldsymbol{\sigma}^{(0)} \to \boldsymbol{\sigma}^{(1)})}{W(\boldsymbol{\sigma}^{(1)} \to \boldsymbol{\sigma}^{(0)})} \cdots \frac{W(\boldsymbol{\sigma}^{(k-1)} \to \boldsymbol{\sigma}^{(k)})}{W(\boldsymbol{\sigma}^{(k)} \to \boldsymbol{\sigma}^{(k-1)})} \tag{10}$$

where $\boldsymbol{\sigma}^{(0)} = \boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}^{(k)} = \boldsymbol{\sigma}(t)$ is the sequence of states occupied by the system over the time interval $[0, t]$ (this is the history of the system between 0 and t). The rates $W(\boldsymbol{\sigma} \to \boldsymbol{\sigma}')$ of hopping from configuration $\boldsymbol{\sigma}$ to configuration $\boldsymbol{\sigma}'$ between t and $t + \mathrm{d}t$ are easily deduced from (3), (4). Inserting the explicit expressions for $W(\boldsymbol{\sigma} \to \boldsymbol{\sigma}')$ leads to

$$\mathcal{Q}_S(t) = -\beta(\mathcal{H}[\boldsymbol{\sigma}(t)] - \mathcal{H}[\boldsymbol{\sigma}(0)]) + \varepsilon Q(t) \tag{11}$$

where we identify Q as the integrated energy current:

$$Q(t) = -\frac{2}{N}\sum_{n=0}^{k}(\pm)\sigma_{j_n}(M_n - \sigma_{j_n}) \tag{12}$$

where σ_{j_n} is the spin being flipped at time n and M_n is the total magnetization at that moment. The sign $+$ $(-)$ corresponds to flipping a spin from group 1 (2). Note that, $\mathcal{H}[\boldsymbol{\sigma}(t)] - \mathcal{H}[\boldsymbol{\sigma}(0)]$ being bounded over time, $\mathcal{Q}_S(t)$ and $\varepsilon Q(t)$ have the same large deviation functions. It is clear that on average,

$$J_\varepsilon = \frac{\langle Q(t)\rangle}{t} = -\frac{2}{N}\left\langle \sum_{j\in 1}\sigma_j(M-\sigma_j)\mathrm{e}^{-\beta_1\sigma_j M/N} - \sum_{j\in 2}\sigma_j(M-\sigma_j)\mathrm{e}^{-\beta_2\sigma_j M/N}\right\rangle. \tag{13}$$

While interpreting $\mathcal{Q}_S(t)$ as an integrated entropy current requires an elaborate reasoning [4], the physical meaning of J_ε as an energy current is much more intuitive. Indeed, the total energy of the system is constant on average in the steady state:

$$\frac{\mathrm{d}\langle\mathcal{H}\rangle}{\mathrm{d}t} = 0 = -(J_1 + J_2) \tag{14}$$

where J_α is the energy flux due to spin flips caused by heat bath α (for instance J_1 is the first term on the rhs of (13)). The quantity $J_\varepsilon = J_1 - J_2$ is therefore a measure of the

J. Stat. Mech. (2005) P02008

energy flowing from group 1 towards group 2. Hence the related entropy current J_S must read

$$J_S = \beta_1 J_1 + \beta_2 J_2 = \varepsilon J_\varepsilon. \tag{15}$$

This interpretation of $\varepsilon J_\varepsilon$ as an entropy current has been discussed, on the grounds of phenomenological thermodynamics, by Rácz and Zia [7]. Note however that there is no such immediate link between the entropy current and an energy current for a system in contact with more than two heat baths.

3. Stationary state distribution

3.1. Van Kampen expansion and Fokker–Planck equation

In this section we derive a Fokker–Planck equation governing the probability $P(x_1, x_2, t)$ of observing the following fluctuations of the spin magnetizations:

$$x_\alpha = \frac{\sum_{j\in\alpha}\sigma_j - Nm_\alpha}{\sqrt{N}}, \qquad \alpha = 1, 2. \tag{16}$$

This is the Van Kampen [11] expansion of the master equation around the mean magnetizations m_α. The $\sqrt{N}$ rescaling is precisely designed for the x_α to have order 1 fluctuations. We find that $P(x_1, x_2, t)$ satisfies the following Fokker–Planck equation:

$$\partial_t P = -\partial_{x_1}\mathcal{J}_1 - \partial_{x_2}\mathcal{J}_2 \tag{17}$$

where the probability current is given by

$$\mathcal{J}_\alpha = f_\alpha(x_1, x_2)P - D_\alpha \partial_{x_\alpha} P. \tag{18}$$

The two-dimensional force (f_1, f_2) does not derive from a potential unless the two heat baths are at the same temperature. The general expression of the force components is

$$\begin{aligned} f_1(x_1, x_2) &= ((\beta_1 - 2)x_1 + \beta_1 x_2)\cosh\beta_1 m - 2\beta_1(x_1 + x_2)m_1 \sinh\beta_1 m \\ f_2(x_1, x_2) &= ((\beta_2 - 2)x_2 + \beta_2 x_1)\cosh\beta_2 m - 2\beta_2(x_1 + x_2)m_2 \sinh\beta_2 m. \end{aligned} \tag{19}$$

The diffusion constants are given by

$$D_\alpha = \cosh\beta_\alpha m - 2m_\alpha \sinh\beta_\alpha m = \sqrt{1 - 4m_\alpha^2}. \tag{20}$$

In the high temperature phase, using that $\beta_{1/2} = \beta \pm \varepsilon$, this may easily be cast in the following form:

$$f_1(x_1, x_2) = -\partial_{x_1}U_\varepsilon + \varepsilon x_2, \qquad f_2(x_1, x_2) = -\partial_{x_2}U_\varepsilon - \varepsilon x_1 \tag{21}$$

where the potential energy has the expression

$$U_\varepsilon(x_1, x_2) = (1-\beta)\frac{(x_1 + x_2)^2}{2} + \frac{(x_1 - x_2)^2}{2} - \varepsilon\frac{x_1^2 - x_2^2}{2}. \tag{22}$$

J. Stat. Mech. (2005) P02008

In the high temperature phase, to which the ensuing analysis will be confined for simplicity, where $m_1 = m_2 = 0$, we thus have to solve

$$\partial_t P = 0 = -\partial_{x_1} \mathcal{J}_1 - \partial_{x_2} \mathcal{J}_2 \tag{23}$$

where the probability current reduces to

$$\begin{aligned} \mathcal{J}_1 &= ((\beta-2)x_1 + \beta x_2 + \varepsilon(x_1+x_2))P - \partial_{x_1} P, \\ \mathcal{J}_2 &= ((\beta-2)x_2 + \beta x_1 - \varepsilon(x_1+x_2))P - \partial_{x_2} P. \end{aligned} \tag{24}$$

When $\beta_1 = \beta_2 = \beta$, that is in equilibrium, the distribution reads

$$P(x_1, x_2) \sim \exp\left[-(1-\beta)\frac{(x_1+x_2)^2}{2} - \frac{(x_1-x_2)^2}{2}\right]. \tag{25}$$

We may find the exact solution to the Fokker–Planck equation by having the intuition, following [11], that the effective potential U_{eff} defined by

$$P(x_1, x_2) = Z^{-1} \exp(-U_{\text{eff}}) \tag{26}$$

will be quadratic in terms of x_1 and x_2. This is suggested by the force being linear in x_1 and x_2. And indeed this naive assumption leads to the effective potential U_{eff} given by

$$U_{\text{eff}}(x_1, x_2) = \frac{1-\beta}{2}(x_1+x_2)^2 + \frac{2}{4+J^2}(x_1[1-J/2] - x_2[1+J/2])^2 \tag{27}$$

where we have introduced the constant $J \equiv 2\varepsilon/(2-\beta)$. It is then an easy task to compute the mean energy current J_ε:

$$\begin{aligned} J_\varepsilon &= \left\langle \sum_{j\in 1}(-2\sigma_j^z(M^z - \sigma_j^z)/N)\mathrm{e}^{-\beta_1\sigma_j^z M^z/N} - \sum_{j\in 2}(-2\sigma_j^z(M^z - \sigma_j^z)/N)\mathrm{e}^{-\beta_2\sigma_j^z M^z/N} \right\rangle \\ &= 2\varepsilon\langle (x_1+x_2)^2\rangle - 2\langle (x_1^2 - x_2^2)\rangle \\ &= \frac{2\varepsilon}{2-\beta} = J. \end{aligned} \tag{28}$$

In terms of the magnetization fluctuations the solution to the Fokker–Planck equation reads

$$P(x_1, x_2) \sim \exp\left[-\frac{1-\beta}{2}(x_1+x_2)^2 - \frac{2}{4+J^2}(x_1[1-J/2] - x_2[1+J/2])^2\right]. \tag{29}$$

A few comments are in order. In spite of the phase space being only a two-dimensional one, this is just enough to allow for inhomogeneous currents to flow (contrary to the case for a one-dimensional phase space). While the total magnetization has global fluctuations equal to those of a system in equilibrium at β, it may be seen that the magnetization difference between the two spin groups is increased with respect to its equilibrium counterpart in the presence of a current:

$$\langle (x_1-x_2)^2\rangle_{J\neq 0} - \langle (x_1-x_2)^2\rangle_{\text{eq},J=0} = \frac{J^2}{4}\frac{2-\beta}{1-\beta}. \tag{30}$$

This provides an example of a nonequilibrium drive giving rise to an *increase* of fluctuations, rather than to a *decrease* (as is usually noted, e.g. in driven lattice gases [12, 13] and spin chains [14]). We now turn to an analysis of the probability flow lines.

J. Stat. Mech. (2005) P02008

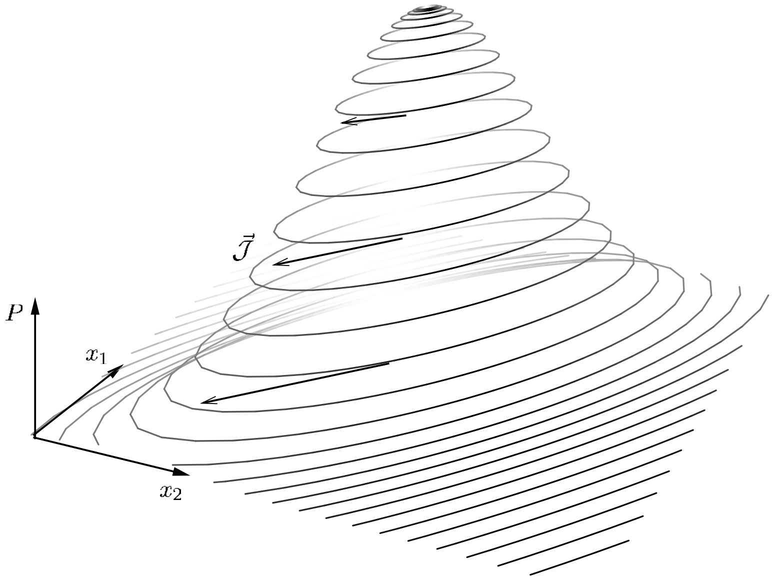

Figure 1. The (x_1, x_2) axes denote the deviations from the group 1 and 2 magnetizations. Flow lines of the probability current $\vec{\mathcal{J}}$ are represented for different values of the constant C in (31). The vertical axis is the probability $P(x_1, x_2)$, illustrating that the flow lines coincide with the isoprobability contours. In this figure, $\beta = 0.8$ and $\varepsilon = 0.7$.

3.2. Flow lines

In equilibrium, by definition, there is no probability current, while in a NESS there are steady (probability) currents. The flow lines, namely the sets of points (x_1, x_2) such that $J_1(x_1, x_2)\,\mathrm{d}x_2 - J_2(x_1, x_2)\,\mathrm{d}x_1 = 0$, in phase space turn out to be ellipses, as shown in figure 1. For $J \neq 0$ the flow lines are ellipses of equation

$$(1-\beta)\left(1+\frac{J^2}{4}\right)(x_1+x_2)^2 + (x_1[1-J/2] - x_2[1+J/2])^2 = C^2 \tag{31}$$

and they coincide with the isoprobability contours, a generic property of linear force driven systems. For an equilibrium system the flow lines can be seen to collapse onto a single point. Interestingly, the shape of the flow lines can be used to infer properties of the steady state distribution: the departure from the ellipses will indicate deviations from linear forces in the Fokker–Planck equation.

3.3. Master equation and effective free energy

We characterize a *state* of our system by the 'local' magnetizations $M_\alpha = \sum_{j\in\alpha} \sigma_j$ and we denote it (in this section only) by $m_\alpha = M_\alpha/N$. The steady state solution to the master equation being denoted by $P_{\mathrm{st}}(M_1, M_2)$, we define the effective free energy $f(m_1, m_2)$ by

$$f(m_1, m_2) = -\lim_{N\to\infty} \frac{\ln P_{\mathrm{st}}(Nm_1, Nm_2)}{N}. \tag{32}$$

J. Stat. Mech. (2005) P02008

We split f into an entropic contribution $s(m_1, m_2)$ and an effective energy $e(m_1, m_2)$. Whether in equilibrium or in a NESS, the entropic part is defined by the combinatoric factor for the number of configurations with M_1 and M_2:

$$\begin{aligned} s(m_1, m_2) &= \frac{1}{N} \ln \begin{pmatrix} \frac{N}{2} \\ \frac{N+2M_1}{4} \end{pmatrix} \begin{pmatrix} \frac{N}{2} \\ \frac{N+2M_2}{4} \end{pmatrix} \\ &= -\sum_\alpha \left[\frac{1+2m_\alpha}{4} \ln \frac{1+2m_\alpha}{4} + \frac{1-2m_\alpha}{4} \ln \frac{1-2m_\alpha}{4} \right]. \end{aligned} \tag{33}$$

This fully defines $e = f + s$. In equilibrium we have that $e_{\mathrm{eq}}(m_1, m_2) = -(\beta/2)(m_1+m_2)^2$, and we wish to find how the nonequilibrium drive modifies this result, namely what kinds of effective interactions between the two groups of spins it generates. Since $\sim\mathrm{e}^{N(s-e)}$ is a stationary solution to the master equation governed by the rates (3), (4), we find that $e(m_1, m_2)$ is a solution to

$$\begin{aligned} 0 = (1-2m_1)&(\mathrm{e}^{-\beta_1(m_1+m_2)-2\partial_1 e} - \mathrm{e}^{+\beta_1(m_1+m_2)}) \\ &+ (1+2m_1)(\mathrm{e}^{+\beta_1(m_1+m_2)+2\partial_1 e} - \mathrm{e}^{-\beta_1(m_1+m_2)}) \\ &+ (1-2m_2)(\mathrm{e}^{-\beta_2(m_1+m_2)-2\partial_2 e} - \mathrm{e}^{+\beta_2(m_1+m_2)}) \\ &+ (1+2m_2)(\mathrm{e}^{+\beta_2(m_1+m_2)+2\partial_2 e} - \mathrm{e}^{-\beta_2(m_1+m_2)}). \end{aligned} \tag{34}$$

To first order in ε one may verify that

$$e(m_1, m_2) = -\frac{1}{2}\beta m^2 - \frac{\varepsilon}{2-\beta}(m_1 - m_2)h(m) + \mathcal{O}(\varepsilon^2), \qquad m = m_1 + m_2 \tag{35}$$

where the function h is the solution to the following first-order ordinary differential equation:

$$(m - \tanh \beta m)\frac{\mathrm{d}h}{\mathrm{d}m} + h(m) - (2-\beta)m = 0, \qquad h(0) = 0. \tag{36}$$

For instance, as $m \to 0$,

$$h(m) = m - \frac{\beta^3}{3(1+3(1-\beta))}m^3 + \mathcal{O}(m^5); \tag{37}$$

thus we recover the leading term of the high temperature Van Kampen expansion. At this stage we have completed our description of the steady state properties. Note that the structure of (35) has flavours of the much more complex one found by Derrida *et al* [15] in the framework of the asymmetric exclusion process for the effective free energy of a given density profile.

4. Energy current distribution

This section is devoted to determining the large deviation function of the time integrated energy current.

4.1. Modified master equation

Let $Q(t)$ be the fluctuating energy current integrated over the time interval $[0, t]$ as defined in (12). We are interested in $p(Q, t)$, the probability that $Q(t) = Q$ at time t, or in its generating function $\hat{p}(\lambda, t) = \langle \mathrm{e}^{-\lambda Q} \rangle$. It is possible to write a master equation for

$$P(M_1, M_2, Q, t) = \mathrm{Prob}\left\{ \sum_{j\in 1} \sigma_j = M_1 \text{ and } \sum_{j\in 2} \sigma_j = M_2 \text{ and } Q(t) = Q \right\}. \tag{38}$$

Inserting the explicit expressions of the transition rates we arrive at

$$\begin{aligned} \partial_t P = {} & \frac{N/2 + M_1 + 2}{N} \mathrm{e}^{-\beta_1 (M+2)/N} P\left(M_1 + 2, M_2, Q - 2\frac{M+1}{N}, t \right) \\ & - \frac{N/2 - M_1}{N} \mathrm{e}^{+\beta_1 M/N} P(M_1, M_2, Q, t) \\ & + \frac{N/2 - M_1 + 2}{N} \mathrm{e}^{+\beta_1 (M-2)/N} P\left(M_1 - 2, M_2, Q + 2\frac{M-1}{N}, t \right) \\ & - \frac{N/2 + M_1}{N} \mathrm{e}^{-\beta_1 M/N} P(M_1, M_2, Q, t) \\ & + \frac{N/2 + M_2 + 2}{N} \mathrm{e}^{-\beta_2 (M+2)/N} P\left(M_1, M_2 + 2, Q - 2\frac{M+1}{N}, t \right) \\ & - \frac{N/2 - M_2}{N} \mathrm{e}^{+\beta_2 M/N} P(M_1, M_2, Q, t) \\ & + \frac{N/2 - M_2 + 2}{N} \mathrm{e}^{+\beta_2 (M-2)/N} P\left(M_1, M_2 - 2, Q + 2\frac{M-1}{N}, t \right) \\ & - \frac{N/2 + M_2}{N} \mathrm{e}^{-\beta_2 M/N} P(M_1, M_2, Q, t). \end{aligned} \tag{39}$$

Going to the generating function

$$\hat{P}(M_1, M_2, \lambda, t) = \sum_{Q} \mathrm{e}^{-\lambda Q} P(M_1, M_2, Q, t) \tag{40}$$

and setting $|\Psi(\lambda, t)\rangle = \sum_{M_1, M_2} \hat{P}(M_1, M_2, \lambda, t) |M_1, M_2\rangle$, we may rewrite equation (39) in the following form:

$$\frac{\mathrm{d}|\Psi(\lambda, t)\rangle}{\mathrm{d}t} = -\hat{H}(\lambda) |\Psi(\lambda, t)\rangle \tag{41}$$

where the operator $\hat{H}(\lambda)$ reads

$$\hat{H}(\lambda) = \sum_{j\in 1} (1 - \sigma_j^x \mathrm{e}^{+2\lambda \sigma_j^z (M^z - \sigma_j^z)/N}) \mathrm{e}^{-\beta_1 \sigma_j^z M^z/N} + \sum_{j\in 2} (1 - \sigma_j^x \mathrm{e}^{-2\lambda \sigma_j^z (M^z - \sigma_j^z)/N}) \mathrm{e}^{-\beta_2 \sigma_j^z M^z/N}. \tag{42}$$

The asymptotic behaviour of

$$\begin{aligned} \hat{p}(\lambda, t) &= \sum_{M_1, M_2} \hat{P}(M_1, M_2, \lambda, t) = \langle \mathbf{p} | \mathrm{e}^{-\hat{H}(\lambda) t} | \Psi(0) \rangle, \\ \langle \mathbf{p} | &= \sum_{M_1, M_2} \langle M_1, M_2 | = \text{projection state} \end{aligned} \tag{43}$$

will be governed by the largest eigenvalue $\mu(\lambda)$ of $-\hat{H}(\lambda)$ in the sense that $\lim_{t\to\infty}(1/t)\ln\hat{p}(\lambda,t) = \mu(\lambda)$, which we now set out to determine. Before embarking into technicalities it is convenient, but by no means compulsory, to perform a similitude transformation on $\hat{H}(\lambda)$:

$$\begin{aligned}\hat{H}_{\rm s}(\lambda) = {\rm e}^{-\beta(M^z)^2/4N}\hat{H}(\lambda){\rm e}^{+\beta(M^z)^2/4N} \\ = \sum_{j\in 1}({\rm e}^{-\beta_1\sigma_j^z M^z/N} - \sigma_j^x{\rm e}^{+(2\lambda-\varepsilon)\sigma_j^z M^z/N-(2\lambda+\beta)/N}) \\ + \sum_{j\in 2}({\rm e}^{-\beta_2\sigma_j^z M^z/N} - \sigma_j^x{\rm e}^{-(2\lambda-\varepsilon)\sigma_j^z M^z/N+(2\lambda-\beta)/N}).\end{aligned} \tag{44}$$

The transformation (44) does not have the same effect as that conducted by Ruijgrok and Tjon [1]—it does not make the resulting operator Hermitian—but it serves the same practical purpose: calculations are performed in a more convenient way where the system symmetries (upon exchanging the roles of 1 and 2) are made obvious. In terms of its symmetrized counterpart $\hat{H}_{\rm s}(\lambda)$, we have that

$$(\hat{H}_{\rm s}(\lambda))^\dagger = \hat{H}_{\rm s}(\varepsilon-\lambda^*). \tag{45}$$

An important consequence of symmetry (45) is that, for λ real, both $\hat{H}_{\rm s}(\lambda)$ and $\hat{H}_{\rm s}(\varepsilon-\lambda)$ have the same spectrum; hence

$$\mu(\lambda) = \mu(\varepsilon-\lambda). \tag{46}$$

This is the Gallavotti–Cohen theorem. A direct consequence for the energy current large deviation function $\pi(q) = (1/t)\ln p(Q = qt,t)$ is that

$$\pi(q) - \pi(-q) = \varepsilon q \tag{47}$$

where we have used that $\pi(q) = \max_\lambda\{\mu(\lambda)+\lambda q\}$. Another useful consequence of (45) is that for $\lambda \in (\varepsilon/2) + {\rm i}\mathbb{R}$, $\hat{H}_{\rm s}(\lambda)$ is Hermitian, which will justify diagonalization in that region of the λ complex plane.

4.2. Mapping to a free boson problem

We introduce, following Ruijgrok and Tjon [1], bosonic operators $a_\alpha, a_\alpha^\dagger$ ($\alpha = 1,2$) to describe magnetizations 1 and 2 in the vicinity of the paramagnetic state:

$$M_\alpha^x = N/2 - 2a_\alpha^\dagger a_\alpha, \qquad M_\alpha^y = -{\rm i}\sqrt{N/2}(a_\alpha^\dagger - a_\alpha), \qquad M_\alpha^z = \sqrt{N/2}(a_\alpha^\dagger + a_\alpha). \tag{48}$$

The relations (48) hold provided we are interested in states such that the number operator $a_\alpha^\dagger a_\alpha$ remain of order unity, that is much smaller than $\sqrt{N}$. In terms of these operators we find that

$$\hat{H}_{\rm s}(\lambda) = \tfrac{1}{2}\,(\,a_1^\dagger \quad a_1 \quad a_2^\dagger \quad a_2\,)\,\Gamma(\lambda)\begin{pmatrix}a_1^\dagger\\ a_1\\ a_2^\dagger\\ a_2\end{pmatrix} + \tfrac{1}{2}(\beta^2 + 4\lambda(\varepsilon-\lambda)) \tag{49}$$

doi:10.1088/1742-5468/2005/02/P02008

J. Stat. Mech. (2005) P02008

with

$$\Gamma = \begin{pmatrix} Z-2\lambda & Z+2 & Z & Z-(2\lambda-\varepsilon) \\ Z+2 & Z+2(\lambda-\varepsilon) & Z+2\lambda-\varepsilon & Z \\ Z & Z+2\lambda-\varepsilon & Z+2\lambda & Z+2 \\ Z-(2\lambda-\varepsilon) & Z & Z+2 & Z-2(\lambda-\varepsilon) \end{pmatrix} \tag{50}$$

where $Z = -\frac{1}{2}\beta(2-\beta)+2\lambda(\varepsilon-\lambda)$. For normalization purposes [16] it is necessary to define an auxiliary matrix $\tilde{\Gamma}$ built from Γ by equating to zero in the latter all matrix elements connecting two creation or two annihilation operators. It is then a simple matter [16] to determine not only the ground state but also the spectrum of $\hat{H}_{\mathrm{s}}(\lambda)$. To do so we introduce the matrix Ω defined as

$$\Omega = \begin{pmatrix} 0 & \mathrm{i}\omega & 0 & 0 \\ -\mathrm{i}\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\omega \\ 0 & 0 & -\mathrm{i}\omega & 0 \end{pmatrix}. \tag{51}$$

We must now evaluate the quantity

$$\mu(\lambda) = \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \ln\frac{\det(\tilde{\Gamma}+\Omega)}{\det(\Gamma+\Omega)} - \frac{1}{2}(\beta^2+4\lambda(\varepsilon-\lambda)). \tag{52}$$

Given that

$$\det(\Gamma+\Omega) = (\omega^2+\omega_+^2)(\omega^2+\omega_-^2) \tag{53}$$

with

$$\omega_\pm(\lambda) = \sqrt{(2-\beta)^2+4\lambda(\varepsilon-\lambda)} \pm \sqrt{\beta^2+4\lambda(\varepsilon-\lambda)} \tag{54}$$

and that similarly

$$\det(\tilde{\Gamma}+\Omega) = (\omega^2+\tilde{\omega}_+^2)(\omega^2+\tilde{\omega}_-^2) \tag{55}$$

with

$$\tilde{\omega}_\pm(\lambda) = \tfrac{1}{2}\Big((2-\beta)^2+4\lambda(\varepsilon-\lambda) \pm \sqrt{[(2-\beta)^2+4\lambda(\varepsilon-\lambda)][\beta^2+4\lambda(\varepsilon-\lambda)]}\Big) \tag{56}$$

we arrive at the following result:

$$\mu(\lambda) = \tfrac{1}{2}(\tilde{\omega}_+ + \tilde{\omega}_- - \omega_+ - \omega_-) - \tfrac{1}{2}(\beta^2+4\lambda(\varepsilon-\lambda)) \tag{57}$$

which simplifies to

$$\mu(\lambda) = 2-\beta-\sqrt{(2-\beta)^2+4\lambda(\varepsilon-\lambda)}. \tag{58}$$

By the same token we obtain the spectrum of $\hat{H}(\lambda)$, whose eigenvalues are given by

$$\mathrm{Sp}(\hat{H}(\lambda)) = \{\omega_+(\lambda)\ell + \omega_-(\lambda)\ell' - \mu(\lambda)\}_{\ell,\ell'\in\mathbb{N}}. \tag{59}$$

Specializing to $\lambda = 0$ we obtain as a side result the spectrum of the master equation evolution operator, whose slowest relaxation time is given by $\omega_+^{-1}(0) = (2(1-\beta))^{-1}$. This again matches the results of [1].

Given that $\pi(q)$ and $\mu(\lambda)$ are the Legendre transforms of each other we arrive at the explicit form of the current large deviation function $\pi(q)$:

$$\pi(q) = \frac{\varepsilon}{2}q + 2 - \beta - \frac{1}{2}\sqrt{(2-\beta)^2+\varepsilon^2}\sqrt{4+q^2} \tag{60}$$

and it has the graph shown in figure 2.

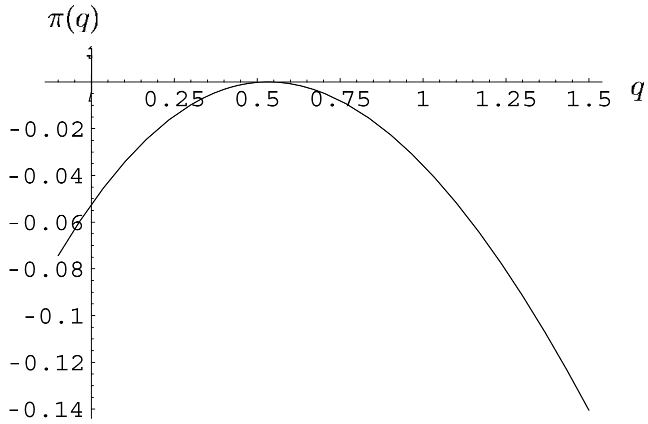

Figure 2. This is the plot of the energy flux large deviation function $\pi(q)$ (vertical axis) as a function of q (horizontal axis) at $\beta = 0.5$ and $\varepsilon = 0.4$.

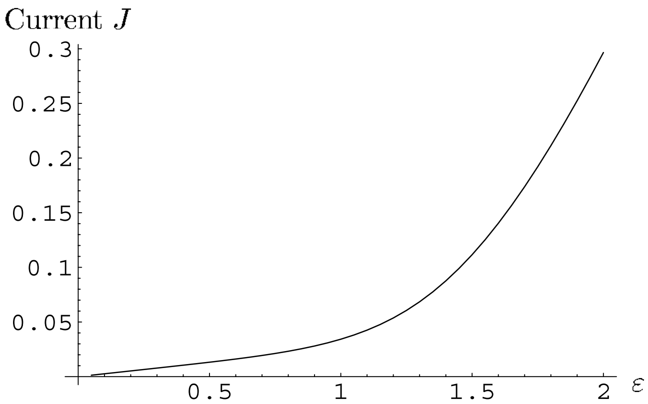

Figure 3. A plot of the average energy current J as a function of $\varepsilon \in [0, \beta]$ at $\beta = 2$ in the ordered phase.

4.3. Energy current in the low temperature phase

Similar methods allowed us to express the generating function of the cumulants of $Q(t)$ in the low temperature phase, at $\beta \geq 1$. The final result is

$$\mu(\lambda) = c_1 + c_2 - \frac{1}{2}\left(\frac{\beta_1}{c_1} + \frac{\beta_2}{c_2}\right) - \sqrt{\left[c_1 + c_2 - \frac{1}{2}\left(\frac{\beta_1}{c_1} + \frac{\beta_2}{c_2}\right)\right]^2 + \frac{4}{c_1 c_2}\lambda(\varepsilon - \lambda)} \qquad (61)$$

where $c_\alpha = \cosh \beta_\alpha (m_1 + m_2) = 1/\sqrt{1 - 4m_\alpha^2}$ and where m_α is the stationary solution of (7), and this is a function of β and ε. However the current is not the order parameter of the phase transition; therefore nothing dramatic is expected to occur for $\mu(\lambda)$ at $\beta = 1$. An important difference from the high temperature result must be emphasized: in the low temperature ordered regime the current is a nonlinear function of ε, as plotted in figure 3. The similarity of the mathematical structures of $\mu(\lambda)$ in the high and low temperature phases seems to be generically related to Langevin equations with linear forces [17].

J. Stat. Mech. (2005) P02008

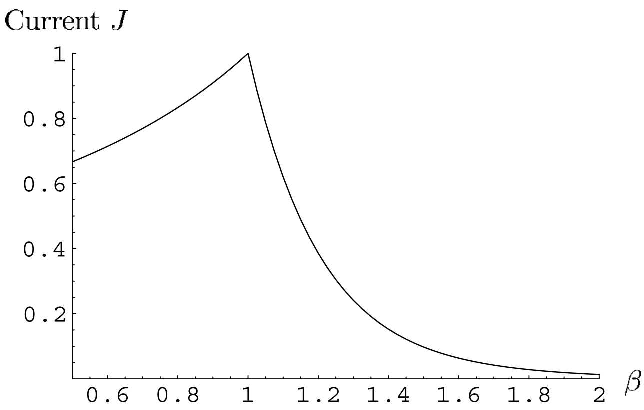

Figure 4. A plot of the average energy current J as a function of $\beta \in [0.5, 2]$ at $\varepsilon = 0.5$.

The energy current at fixed drive $\varepsilon = 0.5$ as a function of $\beta \in [0.5, 2]$ represented in figure 4 shows that, from the disordered to the ordered phase, the current remains finite and continuous, though it develops a cusp at the critical point $\beta = 1$.

4.4. Green–Kubo relations

Exploiting the explicit formula (58) for $\mu(\lambda)$ we find, after differentiation with respect to λ once and twice, that

$$\frac{\langle Q \rangle}{t} = J = \frac{2\varepsilon}{2-\beta}, \qquad \frac{\langle Q^2 \rangle_c}{t} = \frac{4}{2-\beta} + \frac{4}{(2-\beta)^3}\varepsilon^2. \tag{62}$$

Note that defining the diffusion coefficient $D(\beta)$ as the response to an external drive and $\sigma(\beta)$ as the variance of the current fluctuations we find

$$D(\beta) = \left.\frac{\partial J(\beta_1, \beta_2)}{\partial \beta_1}\right|_{\beta_1=\beta_2=\beta} = \frac{1}{2-\beta}, \qquad \sigma(\beta) = \left.\frac{\langle Q^2 \rangle_c}{t}\right|_{\beta_1=\beta_2=\beta}. \tag{63}$$

With these expressions one may verify an integral formulation of the Green–Kubo relation:

$$2\int_{\beta_2}^{\beta_1} \mathrm{d}\beta\, \frac{D(\beta)}{\sigma(\beta)} = \varepsilon. \tag{64}$$

Nevertheless the sole knowledge of $D(\beta)$ and $\sigma(\beta)$ does not give access to the full distribution $\mu(\lambda)$, as opposed to the cases studied by Bodineau and Derrida [5] by means of an additivity principle or by Bertini *et al* [18, 6] who resorted to fluctuating hydrodynamics [18, 6]. In order for the latter approaches to hold, the typical current must scale to zero with the system size at fixed (intensive) external field. This is the second example, aside from the extensively studied asymmetric exclusion process [19], of an interacting system, albeit mean field, in which the entropy (or energy) current can be computed exactly, with the additional property in our case that the Gallavotti–Cohen theorem is fulfilled; hence generalized Green–Kubo relations are satisfied as well.

J. Stat. Mech. (2005) P02008

5. Final comments

We have been able to present explicit and exact results for the steady state of a system made of interacting spins driven far from equilibrium by heat baths at different temperatures. The system described exhibits a ferromagnetic-to-paramagnetic phase transition. The simplicity of some of our results, like Gaussian fluctuations for the magnetizations, are admittedly an artefact of our infinite range, mean field, model. Nevertheless, due to easier mathematics, we have been able to precisely describe the probability flow lines: ellipses in a two-dimensional phase space. Other concepts arising within the framework of dynamical systems theory, like that of topological pressure, once transposed to our model, might equally lend themselves to analytical approaches.

Our other result of interest concerns the computation of an energy current distribution for a system far from equilibrium, that cannot be described by fluctuating hydrodynamics, although it falls within the scope of the Gallavotti–Cohen theorem for Markov processes [4]. To our knowledge, this is the first one of this sort. It is only a first step towards the desirable, but remote, goal of characterizing stationary systems driven very far from equilibrium. It is possible to generalize our results to spins in contact with n thermal baths ($n \ll \sqrt{N}$). The corresponding Fokker–Planck equation remains exactly solvable with the result that the probability flow lines quite notably remain ellipses. The latter now lie within the isoprobability hyperellipsoids. This is also reflected in the entropy current distribution, the structure of which remains unaffected with respect to the $n = 2$ case.

Among the prospects uncovered by the present work, we mention the extension of the urn model of Bena *et al* [20] analysed in the light of the comment by Greenblatt and Lebowitz [21]. In spite of being genuinely nonequilibrium, our model will most probably not display any surprises as far as the Yang–Lee zeros of the partition function are concerned, simply because the phase transition that takes place at $\beta = 1$ is akin to its equilibrium counterpart (and belongs to the same universality class). However the urn model may be very well defined for negative temperatures. Preliminary studies indicate that such rates open the door to limit cycles and chaotic behaviour that we shall explore in future studies.

Acknowledgments

This research was partially supported by the Hungarian Academy of Sciences (Grant No OTKA T043734). It is a pleasure for the authors to thank Henk Hilhorst, Cécile Appert and Bernard Derrida for their comments during the preparation of this work.

References

[1] Ruijgrok Th W and Tjon J A, *Critical slowing down and nonlinear response in an exactly solvable stochastic model*, 1973 *Physica* **65** 539

[2] Gallavotti G and Cohen E G D, *Dynamical ensembles in nonequilibrium statistical mechanics*, 1995 *Phys. Rev. Lett.* **74** 2694 [chao-dyn/9410007]

[3] Kurchan J, *Fluctuation theorem for stochastic dynamics*, 1998 *J. Phys. A: Math. Gen.* **31** 3719 [cond-mat/9709304]

[4] Lebowitz J L and Spohn H, *A Gallavotti–Cohen type symmetry in the large deviation functional for stochastic dynamics*, 1999 *J. Stat. Phys.* **95** 333 [cond-mat/9811220]

[5] Bodineau T and Derrida B, *Current fluctuations in non-equilibrium diffusive systems: an additivity principle*, 2004 *Phys. Rev. Lett.* **92** 180601 [cond-mat/0402305]

[6] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C, *Macroscopic current fluctuations in stochastic lattice gases*, 2004 *Preprint* cond-mat/0407161

[7] Rácz Z and Zia R K P, *Two-temperature kinetic Ising model in one dimension: steady-state correlations in terms of energy and energy flux*, 1994 *Phys. Rev.* E **49** 139

[8] Schmittmann B and Schmüser F, *Exact stationary correlations for a far-from-equilibrium spin chain*, 2002 *Phys. Rev.* E **66** 046130 [cond-mat/0212616]

[9] Hilhorst H J, Schmittmann B and Zia R K P, *Heat transport in the Ising chain*, 2005 in preparation

[10] Garrido P L and Marro J, *Ising models with anisotropic interactions: stationary nonequilibrium states with a nonuniform temperature profile*, 1987 *Physica* A **144** 585

[11] van Kampen N, 1997 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)

[12] Spohn H, *Long range correlations for stochastic lattice gases in a nonequilibrium steady-state*, 1983 *J. Phys. A: Math. Gen.* **16** 4275

[13] van Wijland F and Rácz Z, *Large deviations in weakly interacting boundary driven lattice gases*, 2005 *J. Stat. Phys.* **118** 27 [cond-mat/0404358]

[14] Eisler V, Rácz Z and van Wijland F, *Transverse Ising chain with energy flux: magnetization distributions*, 2003 *Phys. Rev.* E **67** 056129 [cond-mat/0301473]

[15] Derrida B, Lebowitz J L and Speer E R, *Exact large deviation functional of a stationary open driven diffusive system: the asymmetric exclusion process*, 2003 *J. Stat. Phys.* **110** 775 [cond-mat/0205353]

[16] Negele J W and Orland H, 1988 *Quantum Many-Particle Systems* (New York: Addison-Wesley)

[17] Brunet É and Derrida B, 2004 private communication

[18] Bertini L, De Sole A, Gabrielli D, Jona-Lasinio G and Landim C, *Minimum dissipation principle in stationary non-equilibrium steady states*, 2004 *J. Stat. Phys.* **116** 831 [cond-mat/0310072]

[19] Derrida B and Lebowitz J L, *Exact large deviation function in the asymmetric exclusion process*, 1998 *Phys. Rev. Lett.* **80** 209 [cond-mat/9809044]

[20] Bena I, Coppex F, Droz M and Lipowski A, *Yang–Lee zeros for an urn model for the separation of sand*, 2003 *Phys. Rev. Lett.* **91** 160602 [cond-mat/0306739]

[21] Greenblatt R L and Lebowitz J L, *Comment on 'Yang–Lee zeros for an urn model for the separation of sand'*, 2004 *Phys. Rev. Lett.* **93** 238901

J. Stat. Mech. (2005) P02008