

Lyapunov exponents of stochastic systems—from micro to macro

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PAPER

Lyapunov exponents of stochastic systems—from micro to macro

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Abstract. Lyapunov exponents of dynamical systems are defined from the rates of divergence of nearby trajectories. For stochastic systems, one typically assumes that these trajectories are generated under the ‘same noise realization’. The purpose of this work is to critically examine what this expression means. For Brownian particles, we consider two natural interpretations of the noise: intrinsic to the particles or stemming from the fluctuations of the environment. We show how they lead to different distributions of the largest Lyapunov exponent as well as different fluctuating hydrodynamics for the collective density field. We discuss, both at microscopic and macroscopic levels, the limits in which these noise prescriptions become equivalent. We close this paper by providing an estimate of the largest Lyapunov exponent and of its fluctuations for interacting particles evolving with Dean-Kawasaki dynamics.

Keywords: stochastic particle dynamics (theory), connections between chaos and statistical physics

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Contents

1. Introduction	3
2. Lyapunov exponent of a Brownian particle	4
2.1. Noise on the particle	4
2.2. Noise on the environment	4
2.2.1. Particle in a Gaussian random field	4
2.2.2. Calculation of the Lyapunov exponent	5
3. Collective dynamics of Brownian particles	7
3.1. Noise on the particles	8
3.2. Noise on the environment	9
3.3. Comparison between particle-based and environment-based collective dynamics	10
4. Tangent dynamics of the collective density mode	11
4.1. Linearizing the fluctuating hydrodynamics	11
4.1.1. Noise on the particles	12
4.1.2. Noise on the environment	12
4.2. Starting from microscopic dynamics	12
4.2.1. Noise on the particles	12
4.2.2. Noise on the environment	13
4.2.3. Comparison between particle-based and environment-based tangent dynamics	14
4.2.4. From tangent dynamics to Lyapunov exponents	16
5. The Lyapunov exponent of the Dean-Kawasaki equation	17
5.1. The largest Lyapunov exponent	17
5.2. Mean field	18
5.3. Gaussian fluctuations	19
6. Conclusion	21
Appendix A. Time discretization for the environment-based noise	22
Appendix B. Stratonovitch calculus	23
Appendix C. Tangent dynamics of particle-based noise: hydrodynamics derivation versus linearization	24
References	27

1. Introduction

For a given deterministic dynamical system, Lyapunov exponents tell us about the rate at which two copies of the system prepared with close-by initial conditions exponentially diverge from each other in the course of time. The latter rate need not be identical along all phase space directions, hence the existence of a whole spectrum of exponents. Extending the notion of Lyapunov exponents to systems evolving under the action of some external noise has already been carried out some time ago [1–5]. One possible approach, which we henceforth adopt, is to view the noise as an external perturbation that acts in an identical way on the infinitesimally close realizations of the system. The purpose of this article is to critically examine this definition of the Lyapunov exponents (and in particular when applied to the largest one) for systems endowed with stochastic dynamics. A number of ambiguities are still pending, and they boil down to exactly what we mean by ‘same realization of the noise’.

Let us now define the largest Lyapunov exponent. We consider a system described by the vector x which evolves with the equation

$$\dot{x} = f(x). \quad (1)$$

We consider now two copies x_A and x_B of this system, which both evolve with the equation (1). If f is stochastic, both copies evolve with the same noise realization. We suppose the difference between the two copies is small, so the difference $u = x_A - x_B$ evolves according to

$$\dot{u} = J_f(x) u \quad (2)$$

where $J_f(x)$ is the Jacobian matrix of f in x . The (largest) Lyapunov exponent is defined by

$$\lambda(t) = \frac{1}{t} \ln \frac{\|u(t)\|}{\|u(0)\|} \quad (3)$$

where $\|\cdot\|$ stands for the L^2 -norm.

With this definition, we will show in section 2 that taking the ‘same noise realization’ for the two nearby copies x_A and x_B remains ambiguous, even for simple Brownian particles. We will consider two different interpretations of the noise entering our stochastic modelling. In the first case, we will consider a noise intrinsic to each particle, as was frequently done in the literature [1], which we refer to as ‘particle-based noise’. Considering the same noise realization then means that x_A and x_B experience at time t the same noise, say $\eta(t)$, in both realizations, independently of their positions. Then, we will introduce an ‘environment-based noise’, in which we assume the origin of the noise to lie solely in the statistics of the surrounding fluids. Taking the same noise realizations then means that particles at position x at time t experience the same noise $\chi(x, t)$ in both realizations. As we show in section 2, these two different interpretations lead to different distributions of $\lambda(t)$, even for simple Brownian particles, whereas it is impossible to discriminate between these two prescriptions when looking at a single copy of the system.

We then turn to the study of the collective dynamics of Brownian particles in section 3. Following the approach of Dean [6], we construct the fluctuating hydrodynamics

for the density fields corresponding to the two types of noise. Interestingly, these hydrodynamics are typically different, except in the limit where the spatial correlation length of the environment-based noise is much smaller than the interparticle distance.

Next, we define the largest Lyapunov exponent associated to the collective density field in section 4. To do so, we first verify that linearizing the fluctuating hydrodynamics is equivalent to directly coarse-graining the microscopic tangent dynamics. Again, for the two types of noise to be equivalent, the spatial correlation length of the environment-based noise needs to be much smaller than the interparticle distance. This is, however, not sufficient and one also needs to compare initial conditions that are separated by a distance larger than the noise correlation length when constructing the tangent dynamics.

Sections 2–4 thus allow us to unambiguously define the Lyapunov exponent associated to the collective density field of interacting particles. We then provide in section 5 two estimates of its mean values, one at a purely mean-field level and the other by retaining the Gaussian fluctuations of the density field.

2. Lyapunov exponent of a Brownian particle

A single particle undergoing Brownian diffusion in a solvent is arguably the simplest of stochastic systems. In this section, we show that the definition of its Lyapunov exponent, even once the ‘same noise convention’ has been taken, is ambiguous. For sake of simplicity, we consider a one-dimensional system.

2.1. Noise on the particle

The standard description of Brownian motion is to consider the following stochastic differential equation for the position $r(t)$ of a particle:

$$\dot{r} = \eta(t) \quad (4)$$

where $\eta(t)$ is a zero-mean Gaussian white noise the correlations of which satisfy $\langle \eta(t) \eta(t') \rangle = 2D \delta(t - t')$.

To compute the Lyapunov exponent, we consider two copies of our system evolving with the same noise realization. The tangent vector evolves according to the equation:

$$\dot{u} = 0. \quad (5)$$

Since the two particles experience the same noise realization, and since there is nothing else in the system to make their dynamics differ, the distance between them remains trivially constant and $\lambda = 0$.

2.2. Noise on the environment

2.2.1. Particle in a Gaussian random field. For a colloid in a fluid, the origin of the noise lies in the collision with the fluid particles. It is thus natural to consider a (Gaussian) noise *field* $\chi(r, t)$ experienced by a particle at position r at time t . The single-particle dynamics then read

$$\dot{r} = \chi(r(t), t) \quad (6)$$

where $\langle \chi(x, t) \rangle = 0$ and $\langle \chi(x, t) \chi(x', t') \rangle = 2D C(x - x') \delta(t - t')$. The function $C(x)$ represents the spatial correlations of the fluctuations in the fluid. For simplicity, we choose it to be smooth and even (to respect isotropy), which implies $C'(0) = 0$.

In addition, we expect the correlation function $C(x)$ to be maximal at $x = 0$ and to decrease as $|x|$ increases. This implies $C''(0) \leq 0$, which we use to define a characteristic length scale ℓ by $C''(0) \equiv -1/\ell^2$. Furthermore, we normalize $C(0) = 1$ so that the noise amplitude is solely controlled by D . At this point, whenever the function C respects the previous constraints, it is impossible to say if the noise is particle—or environment—based.

Equation (6) can be rewritten as

$$\dot{r} = \int dy \delta(y - r(t)) \chi(y, t). \quad (7)$$

Equation (7) involves a multiplicative noise, which in principle calls for a specification of the time-discretization we resort to. As we show in appendix A, a simplifying mathematical feature makes the discretization an irrelevant feature as long as $C'(0) = 0$. We use the Stratonovich convention in this section, so that the standard rules of differential calculus apply.

Let us now compare equations (4) and (6)–(7). Both equations (4) and (7) have their first Kramers–Moyal coefficient equal to zero (see appendix A) while their second coefficients expectedly coincide:

$$\lim_{\Delta t \rightarrow 0} \frac{\langle [r(t + \Delta t) - r(t)]^2 \rangle}{\Delta t} = 2D \quad (8)$$

Higher order Kramers–Moyal coefficients scale with higher orders of Δt and hence both dynamics lead to the same Fokker–Planck equation. One could thus naively expect their Lyapunov exponents to be equal. This is what we investigate in the next subsection.

2.2.2. Calculation of the Lyapunov exponent. We now consider two infinitesimally close initial conditions, $r_1(0)$ et $r_2(0)$, which evolve with the same noise realization $\chi(y, t)$. The evolution of $u(t) = r_1(t) - r_2(t)$ is given by the linearized (tangent) dynamics

$$\dot{u} = -u(t) \int dy \chi(y, t) \partial_y \delta(y - r(t)) \quad (9)$$

whose solution reads

$$u(t) = u(0) \exp\left[-\int_0^t dt' \int dy \chi(y, t') \partial_y \delta(y - r(t'))\right]. \quad (10)$$

The Lyapunov exponent is thus given by

$$\lambda(t) = -\frac{1}{t} \int_0^t dt' \int dy \chi(y, t') \partial_y \delta(y - r(t')). \quad (11)$$

This is a fluctuating quantity. The global expansion coefficient $\Lambda(t) = t\lambda(t)$ actually satisfies another Langevin equation

$$\dot{\Lambda}(t) = -\int dy \chi(y, t) \partial_y \delta(y - r(t)). \quad (12)$$

To compute the probability distribution of $\lambda(t)$, we will thus write and solve the Fokker–Planck equation satisfied by the joint probability $P(r, \Lambda, t)$. Let us determine the first and second order coefficients of the Kramers–Moyal expansion of the coupled variables Λ and r . As we have chosen a Stratonovich convention, we have

$$\Delta r = \int_t^{t+\Delta t} dt' \int dy \delta(y - r(t) - \frac{1}{2}\Delta r) \chi(y, t') \quad (13)$$

$$\Delta \Lambda = - \int_t^{t+\Delta t} dt' \int dy \partial_y \delta(y - r(t) - \frac{1}{2}\Delta r) \chi(y, t') \quad (14)$$

which allows us to compute

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \Lambda \rangle}{\Delta t} = DC'''(0) = -\frac{D}{\ell^2} \quad (15)$$

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \Lambda^2 \rangle}{\Delta t} = -2DC'''(0) = 2\frac{D}{\ell^2} \quad (16)$$

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \Lambda \Delta r \rangle}{\Delta t} = 2DC'(0) = 0 \quad (17)$$

The correlation between $\Delta \Lambda$ and Δr is zero here, so we actually need not consider the joint probability $P(r, \Lambda, t)$ and we can establish a Fokker–Planck equation for $P(\Lambda, t)$ only:

$$\partial_t P(\Lambda, t) = \frac{D}{\ell^2} (\partial_\Lambda + \partial_\Lambda^2) P(\Lambda, t). \quad (18)$$

We recognize the Fokker–Planck equation of a Brownian particle with position Λ , starting from the origin $\Lambda(0) = 0$ and subject to a constant force, which simply diffuses in the co-moving frame with velocity D/ℓ^2 , and with diffusion constant D/ℓ^2 . We thus have that

$$P(\Lambda, t) = \frac{1}{\sqrt{4\pi D \ell^{-2} t}} e^{-\frac{(\Lambda - D \ell^{-2} t)^2}{4D \ell^{-2} t}}. \quad (19)$$

By changing variable from Λ to $\lambda = \Lambda/t$ we can directly read off the large deviation function of $\Lambda = \lambda t$ as the rate of increase of $P(\lambda, t)$:

$$P(\lambda, t) = \sqrt{\frac{t \ell^2}{4\pi D}} \exp \left[-\frac{t \ell^2}{4D} \left(\lambda + \frac{D}{\ell^2} \right)^2 \right]. \quad (20)$$

The Lyapunov exponent $\lambda(t)$ is thus non zero, not even on average, given that $\langle \lambda \rangle = -D/\ell^2 < 0$. Decreasing the spatial correlations of the noise leads two nearby particles to follow different subsequent trajectories, and it is thus expected that taking ℓ finite allows for fluctuations of λ . It was, however, more difficult to predict that decreasing ℓ was going to decrease the *mean* Lyapunov exponent and make the system

more stable on average: shorter noise correlations could have made both particles pull away from each other.

Note that the probability distribution (20) depends on the spatial correlations of the noise solely through its short-scale structure: it does not depend on the full spatial correlation function but only on $\ell^2 = -1/C''(0)$. In a single particle problem, having a noise ‘attached’ to a particle, as in equation (4), is equivalent to applying the same noise at every location in space, i.e. to $C(x) = 1$ for all x . This in turn implies $\ell = +\infty$ and our two results are then consistent with each other since

$$P(\lambda, t) \underset{\ell \rightarrow +\infty}{\sim} \delta(\lambda). \quad (21)$$

Changing the nature of the noise thus completely modifies the tangent dynamics, even though the two individual dynamics (4) and (6) yield the same Fokker–Planck equation. A prescription which appeared purely philosophical for the dynamics of a single copy of the system turns out to have important consequences when looking at the Lyapunov dynamical stability of the system.

3. Collective dynamics of Brownian particles

Let us now consider N identical particles which we endow with either of the noise prescriptions discussed in the previous section. The local fluctuating density field, defined by $\rho(x, t) = \sum_i \delta(x - r_i(t))$, where $r_i(t)$ denotes the position of particle i at time t , evolves according to some stochastic dynamics.

For particle-based individual noise, this is the well-known Dean-Kawasaki Langevin equation [6] that governs the evolution of ρ , which we briefly rederive in section 3.1. We then construct the fluctuating hydrodynamics stemming from an environment-based noise in section 3.2. Last, we compare the two dynamics in section 3.3.

As we have just shown, the individual dynamics lead to distinctly different tangent dynamics for the two types of noises, except when the correlation length ℓ of the environment-based noise is infinite. It may thus be somewhat of a surprise that the dynamics of the collective local density field become equivalent in the converse limit, when the correlation length ℓ is smaller than the interparticle distance d . It is indeed a nontrivial result that, in the limit $\ell \ll d$, the Dean-Kawasaki equation also governs the evolution of ρ when individual dynamics are driven by a noise field as in (7).

In the general case, when d and ℓ are comparable, the fluctuations of the density field scale differently for the two types of noises. For particle-based noise, the fluctuations of each particle sum up incoherently, and one recovers a noise variance proportional to $\rho(x)$. On the contrary, for environment-based noise, the fluctuations become correlated and the noise variance is thus proportional to $\rho^2(x)$. Our goal here is not to study the general case of environment-based noise with long-range correlations [7–9]. We want two nearby particles to experience independent noises, but we want two particles at the same position in two copies of our system to experience the same noise. This is why we consider a noise field with $\ell \ll d$.

3.1. Noise on the particles

Consider N Brownian particles such that particle j is subjected to a zero-mean Gaussian white noise η_j and evolves according to

$$\dot{r}_j(t) = \eta_j(t). \quad (22)$$

The particles are uncorrelated so that $\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta_{i,j} \delta(t - t')$. We want to determine how the collective mode associated to the density of particles

$$\rho(x, t) = \sum_{j=1}^N \rho_j(x, t) \quad \text{with} \quad \rho_j(x, t) = \delta(x - r_j(t)) \quad (23)$$

evolves. This was done by Dean using Itô calculus [6] and a method that we will follow in the next section. We use a slightly less rigorous path here, which is however more easily generalized. For sake of completeness, we consider the case of a multiplicative noise by allowing D to depend on r_j . We treat ρ as a multi-dimensional function of the $r_j(t)$'s and use Itô lemma to compute its time-evolution¹

$$\dot{\rho}(x, r_j(t)) = \sum_{j=1}^N \dot{r}_j \partial_{r_j} \rho + \sum_{j=1}^N D(r_j) \partial_{r_j}^2 \rho. \quad (24)$$

Using that $\partial_{r_j} \rho = \partial_{r_j} \rho_j$, one gets

$$\dot{\rho}(x, r_j(t)) = \sum_{j=1}^N \partial_{r_j} \delta(x - r_j) \eta_j + \sum_{j=1}^N D(r_j) \partial_{r_j}^2 \delta(x - r_j). \quad (25)$$

We then use that $\partial_{r_j} \delta(x - r_j) = -\partial_x \delta(x - r_j)$ to get

$$\dot{\rho}(x, r_j(t)) = \partial_x \left[-\sum_{j=1}^N \delta(x - r_j) \eta_j + \partial_x \sum_{j=1}^N D(r_j) \delta(x - r_j) \right]. \quad (26)$$

Finally, using that $\sum_{j=1}^N D(r_j) \delta(x - r_j) = \sum_{j=1}^N D(x) \delta(x - r_j) = D(x) \rho(x)$, we find the generalization of Dean's results to multiplicative noise

$$\partial_t \rho(x, t) = \partial_x^2 [D(x) \rho(x, t)] - \partial_x \xi(x, t) \quad \text{with} \quad \xi = \sum_{j=1}^N \rho_j(x, t) \eta_j(t). \quad (27)$$

The noise $\xi(x, t)$ is a sum of Gaussian noises and hence Gaussian. It is completely characterized by its average, which is 0, and its variance, which is

$$\langle \xi(x, t) \xi(x', t') \rangle = \sum_{i,j=1}^N \rho_i(x) \rho_j(x') \langle \eta_i(t) \eta_j(t') \rangle \quad (28)$$

¹ This is the non-rigorous part since Itô lemma applies for twice-differentiable scalar function, which the ρ_j 's are not.

$$= 2 \sum_{j=1}^N D(r_j) \delta(x - r_j(t)) \delta(x' - r_j(t)) \delta(t - t') \quad (29)$$

$$= 2D(x) \delta(x - x') \delta(t - t') \sum_{j=1}^N \delta(x' - r_j(t)) \quad (30)$$

$$= 2D(x) \rho(x, t) \delta(x - x') \delta(t - t') \quad (31)$$

The evolution of ρ is then self-consistently given by

$$\partial_t \rho(x, t) = \partial_x^2 [D(x) \rho(x, t)] - \partial_x \xi^D(x, t) \text{ with } \xi^D(x, t) = \sqrt{\rho(x, t)} \eta(x, t) \quad (32)$$

where

$$\langle \eta(x, t) \rangle = 0 \quad \text{and} \quad \langle \eta(x, t) \eta(x', t') \rangle = 2D(x) \delta(x - x') \delta(t - t'). \quad (33)$$

From now on, we turn back to the case of constant D . Let us now consider the case of environment-based noise.

3.2. Noise on the environment

Consider now the noise field from section 2.2, resulting from a fluctuating environment. To derive the stochastic dynamics of $\rho(x, t)$, we again make use of the Itô lemma, now generalized to a random field. For completeness, an equivalent derivation using Stratonovich conventions is presented in appendix B. Let us consider N particles whose individual dynamics are

$$\frac{dr_j}{dt} = \chi(r_j(t), t) \quad (34)$$

where $\chi(x, t)$ is the Gaussian random field defined in section 2.2. As shown in appendix A, this equation does not depend on the chosen discretization (Itô or Stratonovich) and in this section we use Itô calculus. As before, we define individual and global densities as

$$\rho_i(x, t) = \delta(x - r_i(t)) \quad (35)$$

$$\rho(x, t) = \sum_i \rho_i(x, t) \quad (36)$$

We could, once again, apply Itô Lemma directly to ρ , but we follow, for completeness, the path of Dean [6] here. Let $f(r_i)$ be an arbitrary function of r_i . The generalization of the Itô lemma to a field of noise tells us that

$$\frac{df(r_i(t))}{dt} = f'(r_i(t)) \frac{dr_i}{dt} + D f''(r_i(t)) \quad (37)$$

where we have used that $C(0) = 1$. Using the definition of ρ_i , this can be rewritten as

$$\frac{df(r_i(t))}{dt} = \int dx \rho_i(x, t) [f'(x)\chi(x, t) + D f''(x)] \quad (38)$$

$$= \int dx f(x) [-\partial_x(\rho_i(x, t) \chi(x, t)) + D \partial_x^2 \rho_i] \quad (39)$$

where the second equality comes from an integration by part. Since

$$f(r_i(t)) = \int dx \rho_i(x, t) f(x) \quad (40)$$

one also has that

$$\frac{df(r_i(t))}{dt} = \int dx f(x) \partial_t \rho_i. \quad (41)$$

Since equations (39) and (41) hold for any function f , they yield

$$\partial_t \rho_i(x, t) = D \partial_x^2 \rho_i(x, t) - \partial_x [\rho_i(x, t) \chi(x, t)]. \quad (42)$$

Summing over i and using the definition of the density field we obtain

$$\partial_t \rho(x, t) = D \partial_x^2 \rho(x, t) - \partial_x \Gamma(x, t) \quad \text{with} \quad \Gamma(x, t) = \rho(x, t) \chi(x, t). \quad (43)$$

An interesting feature of this calculation is that the noise acting on ρ is directly given as a functional of ρ , without any prior reference to the individual microscopic densities $\delta(x - r_j)$ (as opposed to equation (27)).

3.3. Comparison between particle-based and environment-based collective dynamics

Without further physical requirements, the two breeds of noise we have introduced do not lead to the same physics for the collective modes, along the same lines as those we have already commented on regarding the particles' Lyapunov exponents. For the density field, this can be seen by inspecting the noise variance in equations (32) and (43).

When each particle has its own noise, the variance of the collective noise ξ is proportional to ρ . This simply comes from the fact that two nearby particles remain independent. Therefore, their contributions to the density fluctuations add incoherently. When particles experience a Gaussian field χ with finite correlation length, nearby particles are no longer independent: density fluctuations are due to the environment and their amplitude scales with the local density ρ . The noise variance is thus proportional to ρ^2 .

Note that taking the limit $\ell \rightarrow 0$ in (43), using² $C(x) = \delta(x)/\delta(0)$, leads to

$$\langle \Gamma(x, t) \Gamma(x', t') \rangle = \frac{2D}{\delta(0)} \rho(x, t)^2 \delta(x - x') \delta(t - t') \quad (44)$$

This does not suffice to recover the noise of independent particles since two point-like particles at the same position x still experience the same noise.

To properly recover the result of the noise on particles, we have to introduce the typical size of the particles d . When it is larger than the typical correlation length of

² The $\delta(0)$ is present for normalization purpose.

the noise ℓ , we expect to recover an independent contribution from each particle to the fluctuations of the local density field. Since two particles cannot be correlated, because $\|r_i - r_j\| > d \gg \ell$, the spatial correlation function reduces to

$$C(r_i(t) - r_j(t')) = \delta_{i,j} C(0) = \delta_{i,j} \quad (45)$$

and the correlations of the noise are then given by

$$\begin{aligned} \langle \Gamma(x, t) \Gamma(x', t') \rangle &= 2D \delta(t - t') \sum_{i,j=1}^N \delta(x - r_i) \delta(x' - r_j) C(r_j - r_i) \\ &= 2D \delta(t - t') \sum_{i,j=1}^N \delta(x - r_i) \delta(x' - r_j) \delta_{i,j} \\ &= 2D \delta(t - t') \sum_{i=1}^N \delta(x - r_i) \delta(x' - x) \\ &= 2D \rho(x, t) \delta(x' - x) \delta(t - t') \end{aligned} \quad (46)$$

which is what we obtained with the noise on particles.

To summarize, particle-based and environment-based noises yield two different collective dynamics, mostly because in the latter case nearby particles are effectively correlated. The proper way to recover uncorrelated noise for the environment-based noise is thus to consider particles with finite-size d and take $d \gg \ell$. Note that this simply amounts to considering cases where $\|r_i - r_j\| \gg \ell$, i.e. systems which are dilute at the scale of the fluid's correlation length.

4. Tangent dynamics of the collective density mode

As for the single particle case, we thus expect that the type of noise has an impact on the divergence of nearby trajectories at the macroscopic scale, and hence the Lyapunov exponents associated to the collective density mode ρ . Note that it is also unclear whether the tangent dynamics associated to ρ , obtained by linearizing the stochastic partial differential equation obeyed by $\rho(x, t)$, should coincide with the direct coarse-graining of microscopic tangent dynamics. In this section we thus show that linearizing and projecting on the collective density modes are indeed commuting operations. The approach followed analytically in [10] to compute the fluctuations of the Lyapunov exponent of spatially extended systems incidentally sits on firmer grounds.

4.1. Linearizing the fluctuating hydrodynamics

Let us consider two infinitesimally close initial density profiles $\rho_1(x, 0)$ and $\rho_2(x, 0)$. The evolution of their difference $u(x, t) = \rho_1(x, t) - \rho_2(x, t)$ is obtained by linearizing the equations (32) and (43).

4.1.1. *Noise on the particles.* The tangent evolution associated to (32) is

$$\partial_t u(x, t) = D \partial_x^2 u(x, t) - \partial_x \xi_u^D(x, t) \text{ with } \xi_u^D(x, t) = \frac{u(x, t)}{2\sqrt{\rho(x, t)}} \eta(x, t) \quad (47)$$

where $\eta(x, t)$ is the zero-mean Gaussian white noise defined in (32).

4.1.2. *Noise on the environment.* The tangent evolution associated to (43) is

$$\partial_t u(x, t) = D \partial_x^2 u(x, t) - \partial_x \Gamma_u^D(x, t) \text{ with } \Gamma_u^D(x, t) = u(x, t) \chi(x, t) \quad (48)$$

and $\chi(x, t)$ is the zero-mean Gaussian white noise defined in (6).

4.2. Starting from microscopic dynamics

Let us now start from the microscopic dynamics and derive the hydrodynamic behavior of the differences between two copies of the system. We take two sets $\{r_j\}$ and $\{r_j + \delta r_j\}$ of infinitely close initial positions evolving with the same equation of evolution. We can then define two densities:

$$\rho(x, t) = \sum_{j=1}^N \delta(x - r_j(t)) \quad (49)$$

$$\tilde{\rho}(x, t) = \sum_{j=1}^N \delta(x - r_j(t) - \delta r_j(t)) \quad (50)$$

We want to establish the evolution equation of the (small) differences between the two density fields

$$u(x, t) = \tilde{\rho}(x, t) - \rho(x, t) \simeq - \sum_{j=1}^N \delta r_j(t) \partial_x \delta(x - r_j(t)). \quad (51)$$

4.2.1. *Noise on the particles.* For the noise on particles, the positions and perturbations evolve according to

$$\frac{d\delta r_j}{dt} = 0 \quad (52)$$

which means that the δr_j are constant. Following the path of section 3.1, we now apply the Itô lemma to u:

$$\frac{\partial u}{\partial t} = \sum_{j=1}^N \frac{\partial u}{\partial r_j} \eta_j + D \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 u}{\partial r_i \partial r_j} \delta_{i,j} \quad (53)$$

$$= \sum_{j=1}^N \partial_x^2 \delta r_j \delta(x - r_j) \eta_j + D \partial_x^2 u \quad (54)$$

which can be rewritten as

$$\partial_t u = D \partial_x^2 u + \partial_x \xi_u(x, t) \quad \text{with} \quad \xi_u = \partial_x \sum_{j=1}^N \delta r_j \delta(x - r_j) \eta_j \quad (55)$$

and hence

$$\langle \xi_u(x, t) \rangle = 0 \quad (56)$$

$$\langle \xi_u(x, t) \xi_u(x', t') \rangle = 2D \delta(t - t') \partial_x \partial_{x'} \sum_{j=1}^N \delta r_j^2 \delta(x - r_j) \delta(x - x'). \quad (57)$$

Let us now compare this noise with the noise $\xi_u^D = \frac{u}{2\sqrt{\rho}} \eta$ obtained by linearizing the fluctuating hydrodynamics in equation (47). The latter satisfies

$$\begin{aligned} \langle \xi_u^D(x, t) \xi_u^D(x', t') \rangle &= \frac{D u(x, t) u(x', t')}{2\sqrt{\rho(x, t)\rho(x', t')}} \delta(x - x') \delta(t - t') \\ &= \frac{D \partial_x \partial_{x'} \sum_{i,j=1}^N \delta r_i \delta r_j \delta(x - r_i) \delta(x' - r_j)}{2\sqrt{\sum_{k=1}^N \delta(x - r_k) \sum_{n=1}^N \delta(x' - r_n)}} \delta(x - x') \delta(t - t'). \end{aligned}$$

At first glance, this looks different from (57). As shown in appendix C, these two variances are, however, equivalent. This can be seen by noting that the noise ξ^D appearing in the fluctuating hydrodynamics is equivalent to ξ , i.e.

$$\sqrt{\rho(x, t)} \eta(x, t) = \sum_{i=1}^N \delta(x - r_i(t)) \eta_i(t). \quad (58)$$

Linearizing this equality with respect to r_i then gives

$$\xi_u^D(x, t) = \frac{u(x, t)}{2\sqrt{\rho(x, t)}} \eta(x, t) = \sum_{i=1}^N \delta r_i \partial_x \delta(x - r_i(t)) \eta_i(t) = \xi_u(x, t). \quad (59)$$

At this stage, we have thus shown that linearizing the fluctuating hydrodynamics or coarse-graining the microscopic tangent dynamics yields an equivalent equation of evolution for the tangent field $u(x, t)$ for the particle-based noise.

4.2.2. Noise on the environment. Let us start by establishing the evolution of the δr_j , defined in equation (49), linearizing the microscopic equation (6):

$$\frac{d\delta r_j}{dt} = - \int dy \chi(y, t) \delta r_j(t) \partial_y \delta(y - r_j(t)) \quad (60)$$

$$= \delta r_j(t) \int dy \partial_y \xi(y) \delta(y - r_j(t)) \quad (61)$$

$$= \delta r_j(t) \chi'(r_j(t)). \quad (62)$$

Itô lemma applied to the tangent field $u(x, t) = -\sum_{j=1}^N \delta r_j \partial_x \delta(x - r_j(t))$ defined in equation (51) then yields:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{j=1}^N \frac{\partial u}{\partial r_j} \chi(r_j) + \sum_{j=1}^N \frac{\partial u}{\partial \delta r_j} \delta r_j \chi'(r_j) + D \sum_{i,j=1}^N \frac{\partial^2 u}{\partial r_i \partial r_j} C(r_i - r_j) \\ &+ D \sum_{i,j=1}^N \frac{\partial^2 u}{\partial r_i \partial \delta r_j} \delta r_j C'(r_j - r_i) - D \sum_{i,j=1}^N \frac{\partial^2 u}{\partial \delta r_i \partial \delta r_j} \delta r_i \delta r_j C''(r_i - r_j). \end{aligned} \quad (63)$$

Let us now show how this equation can be greatly simplified. First, since u is linear in the δr_j 's, we have that $\partial_{\delta r_j, \delta r_i} u = 0$. Furthermore, since $\partial_{r_i, \delta r_j} u \propto \delta_{i,j}$ then

$$D \sum_{i,j=1}^N \frac{\partial^2 u}{\partial r_i \partial \delta r_j} \delta r_j C'(r_j - r_i) \propto C'(0) = 0. \quad (64)$$

Moreover,

$$D \sum_{i,j=1}^N \frac{\partial^2 u}{\partial r_i \partial r_j} C(r_i - r_j) = -D \sum_{i,j,k=1}^N \delta r_k \partial_x \partial_{r_i} \partial_{r_j} \delta(x - r_k) C(r_i - r_j) \quad (65)$$

$$= -D \sum_{k=1}^N \delta r_k \partial_x^3 \delta(x - r_k) C(0) = D \partial_x^2 u. \quad (66)$$

Finally, let us consider the two first terms of equation (63), which can be factorized as:

$$\sum_{j=1}^N \left[\frac{\partial u}{\partial r_j} \chi(r_j) + \frac{\partial u}{\partial \delta r_j} \delta r_j \chi'(r_j) \right] = -\sum_{j=1}^N \delta r_j \partial_{r_j} [\partial_x \delta(x - r_j) \chi(r_j)]. \quad (67)$$

We can then use the fact that $\partial_x [\delta(x - r_j)] \chi(r_j) = \partial_x [\delta(x - r_j) \chi(x)]$ to get

$$-\partial_x [\chi(x) \sum_{j=1}^N \delta r_j \partial_{r_j} \delta(x - r_j)]. \quad (68)$$

Finally, we use again that $\partial_{r_j} \delta(x - r_j) = -\partial_x \delta(x - r_j)$ to obtain the simple form

$$\partial_x [\chi(x) \sum_{j=1}^N \delta r_j \partial_x \delta(x - r_j)] = -\partial_x [u(x) \chi(x)]. \quad (69)$$

All in all, equation (63) simplifies into

$$\partial_t u(x, t) = D \partial_x^2 u(x, t) - \partial_x \Gamma_u(x, t) \quad \text{with} \quad \Gamma_u(x, t) = u(x, t) \chi(x, t) \quad (70)$$

which is exactly the same equation as (48), obtained by linearizing the fluctuating hydrodynamics (43). In particular, the noises Γ_u and Γ_u^D are identical.

4.2.3. Comparison between particle-based and environment-based tangent dynamics.

Again, we want to show that when the correlation length of the environment ℓ is much shorter than the interparticle distance d , particle-based and environment-based noises become similar. To do so, we thus look at the variance of $\Gamma_u(x, t)$ as $d \gg \ell$.

Starting from equation (70) and using the explicit expression of $u(x, t)$, one gets

$$\begin{aligned} \frac{\langle \Gamma_u(x, t) \Gamma_u(x', t') \rangle}{2D} &= \delta(t - t') u(x, t) u(x', t') C(x - x') \\ &= \delta(t - t') \sum_{i,j=1}^N \delta r_i \delta r_j \partial_x [\delta(x - r_i)] \partial_{x'} [\delta(x' - r_j)] C(x - x'). \end{aligned}$$

Using that $\partial_x [\delta(x - r_i)] = -\partial_{r_i} [\delta(x - r_i)]$, this becomes

$$\begin{aligned} \frac{\langle \Gamma_u(x, t) \Gamma_u(x', t') \rangle}{2D} &= \delta(t - t') \sum_{i,j=1}^N \delta r_i \delta r_j \partial_{r_i} [\delta(x - r_i)] \partial_{r_j} [\delta(x' - r_j)] C(x - x') \\ &= \delta(t - t') \sum_{i,j=1}^N \delta r_i \delta r_j \partial_{r_i} \partial_{r_j} [\delta(x - r_i) \delta(x' - r_j)] C(r_i - r_j). \end{aligned}$$

If $d \gg \ell$, $C(r_i(t) - r_j(t)) = \delta_{i,j} C(0) = \delta_{i,j}$. Then

$$\begin{aligned} \langle \Gamma_u(x, t) \Gamma_u(x', t') \rangle &= 2D \delta(t - t') \left(\sum_{i,j=1}^N \delta r_i \delta r_j \partial_{r_i} \partial_{r_j} [\delta(x - r_i) \delta(x' - r_j) \delta_{i,j}] \right) \\ &= 2D \delta(t - t') \partial_x \partial_{x'} \left[\sum_{i,j=1}^N \delta r_i \delta r_j \delta(x - r_i) \delta(x' - r_j) \delta_{i,j} \right] \\ &= 2D \delta(t - t') \partial_x \partial_{x'} \left[\sum_{j=1}^N \delta r_j^2 \delta(x - r_j) \delta(x - x') \right] \end{aligned} \quad (71)$$

which is the same as the variance $\langle \xi_u(x, t) \xi_u(x', t') \rangle$ of the noise $\xi_u(x, t)$ appearing in the fluctuating ‘tangent’ hydrodynamics (55) stemming from a particle-based noise.

Note that one difference remains between these two cases: for the particle-based noise, the δr_j ’s are constant whereas they evolve for the environment-based noise. Let us now focus on the latter case to understand the underlying physics. The dynamics on δr_j ’s read

$$\delta \dot{r}_j = \chi'(r_j) \delta r_j = \delta r_j \int dy \partial_y \delta(y - r_j) \chi(y) \equiv \delta \chi_j. \quad (72)$$

This is a Langevin equation with a multiplicative noise $\delta \chi_j$. One trivially has $\langle \delta \chi_j \rangle = 0$ while its correlations are given by

$$\langle \delta \chi_j(t) \delta \chi_j(t') \rangle = 2D \delta r_j^2 \delta(t - t') \int dy dz \partial_y [\delta(y - r_j)] \partial_z [\delta(z - r_j)] C(y - z).$$

Integrating by part on y and z then yields

$$\begin{aligned} \langle \delta \chi_j(t) \delta \chi_j(t') \rangle &= -2D \delta r_j^2 \delta(t - t') \int dy dz \delta(y - r_j) \delta(z - r_j) C''(y - z) \\ &= -2D \delta r_j^2 \delta(t - t') C''(0) = 2D \left(\frac{\delta r_j}{\ell} \right)^2 \delta(t - t'). \end{aligned} \quad (73)$$

If one studies linearized dynamics at a scale much shorter than the environment correlation length ($\delta r_j \ll \ell$), the second cumulant of $\delta\chi_j$ vanishes. The noise on δr_j is identically zero and $\delta\dot{r}_j = 0$. Physically, this is consistent with the fact that two nearby particles separated by a distance much shorter than ℓ experience the same noise. Conversely, if one studies tangent dynamics at a scale larger than ℓ , the two initial conditions for particle j lead to two different noise realizations. Particle-based and environment-based noise then yield the same form of tangent fluctuating hydrodynamics, but the underlying δr_j 's have different dynamics.

4.2.4. From tangent dynamics to Lyapunov exponents. In this section we have shown that linearizing the microscopic dynamics and then coarse-graining the corresponding tangent dynamics is equivalent to starting from the coarse-grained dynamics and linearizing it. Again, the equivalence between particle-based and environment-based noises when the correlation length of the environment is much shorter than the interparticle distance is valid. This legitimates the approach to compute the large deviation of the largest Lyapunov exponent in large driven diffusive systems followed in [10] which started directly from the fluctuating hydrodynamics.

To compute the Lyapunov exponent starting from the tangent dynamics, it is often convenient to introduce the normalized tangent field $v \equiv u/\|u\|$. For particle-based noise, the dynamics of v read

$$\begin{aligned} \partial_t v(x, t) = & D \partial_x^2 v(x, t) - \partial_x \left[\frac{v(x, t)}{2\sqrt{\rho(x, t)}} \eta(x, t) \right] \\ & - v(x, t) \int dy \left(v(y, t) D \partial_y^2 v(y, t) - v(y, t) \partial_y \left[\frac{v(y, t)}{2\sqrt{\rho(y, t)}} \eta(y, t) \right] \right) \end{aligned} \quad (74)$$

whereas for the environment-based noise, it reads

$$\begin{aligned} \partial_t v(x, t) = & D \partial_x^2 v(x, t) - \partial_x [v(x, t) \chi(x, t)] \\ & - v(x, t) \int dy (v(y, t) D \partial_y^2 v(y, t) - v(y, t) \partial_y [v(y, t) \chi(y, t)]). \end{aligned} \quad (75)$$

Note that we have used standard differential calculus, and not Itô stochastic calculus, to derive these formulae. This is legitimate, as surprising as this may seem, since Dean's equation and its linearized version are identical in both prescriptions. This has often been asserted in the literature [11] and a detailed proof can be found in appendix C of [12].

The largest Lyapunov exponent is then given by

$$\lambda_{\text{part}}(t) = \frac{1}{t} \int_0^t dt' \int dx \left(v(x, t') D \partial_x^2 v(x, t') - v(x, t') \partial_x \left[\frac{v(x, t')}{2\sqrt{\rho(x, t')}} \eta(x, t') \right] \right)$$

and

$$\lambda_{\text{env}}(t) = \frac{1}{t} \int_0^t dt' \int dx (v(x, t') D \partial_x^2 v(x, t') - v(x, t') \partial_x [v(x, t') \chi(x, t')])$$

The cumulant generating function corresponding to these Lyapunov exponents have been computed up to fifth order in [10]. Interestingly, the particle-based noise corresponds to free particles, as expected, while the Kipnis–Marchioro–Presutti [13] model corresponds to a realization of the environment-based noise.

5. The Lyapunov exponent of the Dean-Kawasaki equation

5.1. The largest Lyapunov exponent

Let us now discuss how the approach presented above could be used to compute the fluctuations of Lyapunov exponents for interacting systems. We consider the fluctuating hydrodynamics of N particles interacting via a pair-potential V , as derived by Dean [6]:

$$\begin{aligned} \frac{\partial \rho(\mathbf{x}, t)}{\partial t} = & D \nabla_{\mathbf{x}}^2 \rho(\mathbf{x}, t) + \nabla_{\mathbf{x}}(\sqrt{\rho(\mathbf{x}, t)} \boldsymbol{\eta}(\mathbf{x}, t)) \\ & + \nabla_{\mathbf{x}} \left[\int d\mathbf{y} \rho(\mathbf{x}, t) \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t) \right] \end{aligned} \quad (76)$$

where $\boldsymbol{\eta}(\mathbf{x}, t)$ is a Gaussian white noise of variance $2D$. As shown above, this corresponds to particle-based noise as much as to an environment-based noise in the proper limit. The tangent vector u thus evolves according to

$$\begin{aligned} \partial_t u(\mathbf{x}, t) = & D \nabla_{\mathbf{x}}^2 u(\mathbf{x}, t) + \nabla_{\mathbf{x}} \left(\frac{u(\mathbf{x}, t)}{2\sqrt{\rho(\mathbf{x}, t)}} \boldsymbol{\eta}(\mathbf{x}, t) \right) \\ & + \nabla_{\mathbf{x}} \left(\int d\mathbf{y} [u(\mathbf{x}, t) \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t) + \rho(\mathbf{x}, t) \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) u(\mathbf{y}, t)] \right) \end{aligned} \quad (77)$$

Again, we introduce a normalized tangent vector $v = u/\|u\|$, the dynamics of which are given by

$$\begin{aligned} \partial_t v(\mathbf{x}, t) = & D \nabla_{\mathbf{x}}^2 v(\mathbf{x}, t) + \nabla_{\mathbf{x}} \left(\frac{v(\mathbf{x}, t)}{2\sqrt{\rho(\mathbf{x}, t)}} \boldsymbol{\eta}(\mathbf{x}, t) \right) \\ & + \nabla_{\mathbf{x}} \left(\int d\mathbf{y} [v(\mathbf{x}, t) \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t) + \rho(\mathbf{x}, t) \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) v(\mathbf{y}, t)] \right) \\ & - v(\mathbf{x}, t) \int d\mathbf{z} \left\{ D v(\mathbf{z}, t) \nabla_{\mathbf{z}}^2 v(\mathbf{z}, t) + v(\mathbf{z}, t) \nabla_{\mathbf{z}} \left[\frac{v(\mathbf{z}, t)}{2\sqrt{\rho(\mathbf{z}, t)}} \boldsymbol{\eta}(\mathbf{z}, t) \right] \right. \\ & \left. + v(\mathbf{z}, t) \nabla_{\mathbf{z}} \left(\int d\mathbf{y} [v(\mathbf{z}, t) \nabla_{\mathbf{z}} V(\mathbf{z} - \mathbf{y}) \rho(\mathbf{y}, t) + \rho(\mathbf{z}, t) \nabla_{\mathbf{z}} V(\mathbf{z} - \mathbf{y}) v(\mathbf{y}, t)] \right) \right\} \end{aligned}$$

meaning that the largest Lyapunov exponent is given by

$$\lambda(t) = \frac{1}{t} \int_0^t dt' \int dx \left\{ D v(\mathbf{x}, t') \nabla_{\mathbf{x}}^2 v(\mathbf{x}, t') + v(\mathbf{x}, t') \nabla_{\mathbf{x}} \left[\frac{v(\mathbf{x}, t')}{2\sqrt{\rho(\mathbf{x}, t')}} \boldsymbol{\eta}(\mathbf{x}, t') \right] \right. \\ \left. + v(\mathbf{x}, t') \nabla_{\mathbf{x}} \left(\int dy [v(\mathbf{x}, t') \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t') + \rho(\mathbf{x}, t') \nabla_{\mathbf{x}} V(\mathbf{x} - \mathbf{y}) v(\mathbf{y}, t')] \right) \right\}.$$

This explicit formula for the largest Lyapunov exponent could then be used to compute its cumulant-generating function, following the path set in [10].

5.2. Mean field

The simplest approximation scheme that can be implemented consists in retaining Gaussian fluctuations for the density field ρ and the tangent vector v . This section is devoted to establishing the feasibility of such an approximation. Motivations can be found in the statics of simple classical or quantum fluids, where this is called the random phase approximation (RPA), and we shall keep this name in what follows. Recent works in which this approximation has proved useful within a dynamic framework can be found [14] or [15] (this includes a discussion on the range of validity of the RPA approximation, which becomes exact in specific limiting cases). Here, we linearize the dynamics of both the ρ and v fields, and use this as the simplest possible approximation. It would certainly be interesting to understand whether this corresponds to a limiting case for the tangent dynamics as well. The expansion is carried out in powers of $\psi = \rho - \rho_0$ and $\chi = v - v_0$, where $v_0(\mathbf{x})$ is the normalized tangent field obtained within a straight mean-field approximation. We begin with the evolution equation for the normalized tangent vector v , which reads, with condensed notations:

$$\partial_t v = Av - v \int v \cdot Av \quad (78)$$

where A is a linear operator acting upon v . It is important to note that A functionally depends on the density field ρ and on the external noise $\boldsymbol{\xi}$. We find that

$$Av(\mathbf{x}, t) = \nabla \cdot \left[D \nabla v(\mathbf{x}, t) + \rho(\mathbf{x}, t) \int_{\mathbf{y}} \nabla V(\mathbf{x} - \mathbf{y}) v(\mathbf{y}, t) \right. \\ \left. + v(\mathbf{x}, t) \int_{\mathbf{y}} \nabla V(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, t) + \sqrt{2D} \frac{v(\mathbf{x}, t)}{2\sqrt{\rho}} \boldsymbol{\xi}(\mathbf{x}, t) \right] \quad (79)$$

where $\boldsymbol{\xi} = (2D)^{-1/2} \boldsymbol{\eta}$ is a delta correlated Gaussian white noise. Within a mean-field approximation, the density field ρ is replaced with its uniform average ρ_0 , and the noise is neglected. With these simplifications (80) leads, in Fourier space, to

$$\partial_t v(\mathbf{k}, t) = -\Omega_{\mathbf{k}} v(\mathbf{k}, t) + v(\mathbf{k}, t) \int_{\mathbf{k}'} v(-\mathbf{k}', t) \Omega_{\mathbf{k}'} v(\mathbf{k}', t) \quad (80)$$

where $\Omega_{\mathbf{k}} = Dk^2(1 + \beta\rho_0 V(\mathbf{k}))$ ($\beta = D^{-1}$ is an inverse temperature), and where $V(\mathbf{k})$ is the Fourier transform of the interaction potential, namely $V(\mathbf{k}) = \int_{L^3} d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x})$. We assume the system to be enclosed in a cubic box of linear size L with periodic boundary conditions, hence the inverse Fourier transform is given by $V(\mathbf{x}) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{k})$, with $\int_{\mathbf{k}} = L^{-3} \sum_{\mathbf{k}}$, and the \mathbf{k} 's components are integer multiples of $2\pi/L$. The largest Lyapunov exponent itself is given in terms of the stationary solution $v_0(\mathbf{k})$ of (81) by

$$\lambda_M = - \int_{\mathbf{k}'} v_0(-\mathbf{k}') \Omega_{\mathbf{k}'} v_0(\mathbf{k}'). \quad (81)$$

Using \mathbf{q} to denote the wave vector that minimizes $\Omega_{\mathbf{k}}$, it is immediately apparent that

$$v_0(\mathbf{x}) = W \cos \mathbf{q} \cdot \mathbf{x}, \quad v_0(\mathbf{k}) = W^{-1}(\delta_{\mathbf{k},-\mathbf{q}} + \delta_{\mathbf{k},\mathbf{q}}) \quad (82)$$

where $W = \sqrt{2/L^3}$ is a normalizing factor that ensures $\int_{\mathbf{x}} v_0^2(\mathbf{x}) = 1$. Since we have in mind a smooth potential, like that of interacting harmonic spheres that have been recently used in many studies of glass-formers, and which has

$$V(r) = \varepsilon\theta(\sigma - r)(1 - r/\sigma)^2, \quad V(\mathbf{k}) = 8\pi\varepsilon\sigma^3 \frac{2k\sigma + k\sigma \cos k\sigma - 3 \sin k\sigma}{(k\sigma)^5}, \quad (83)$$

it is clear that $\mathbf{q} = \pm \frac{2\pi}{L} \mathbf{e}$, where \mathbf{e} is either of three basis unit vectors. The resulting largest Lyapunov exponent then reads, in the large system size limit

$$\lambda_M = -\frac{4\pi^2}{L^2} D \left[1 + \beta\rho_0 \frac{2\pi\varepsilon\sigma^3}{15} \right]. \quad (84)$$

We have used this harmonic sphere interaction for concreteness, but our result for $\lambda = -\Omega_{\mathbf{q}}$ is of course more general.

With the reference fields around which to expand now at hand, namely ρ_0 and v_0 , here is how we could set up a Gaussian (RPA) expansion around mean-field. We would start by writing the evolution equation for $\chi = v - v_0$ to linear order in χ , $\psi = \rho - \rho_0$, and in the noise ξ . We now address how the average Lyapunov exponent and its fluctuations get renormalized by quadratic fluctuations around mean-field.

5.3. Gaussian fluctuations

The linearized dynamics for $\chi = v - v_0$ are given by

$$\begin{aligned} \partial_t \chi &= A_0 \chi - \chi \int v_0 A_0 v_0 - v_0 \int (v_0 A_0 \chi + \chi A_0 v_0) \\ &\quad + \delta A v_0 - v_0 \int v_0 \delta A v_0 \end{aligned} \quad (85)$$

where A_0 is the operator A evaluated at zero noise and at $\psi = \rho - \rho_0 = 0$, while δA is the linear correction to A in an expansion in the ψ and ξ fields. The action of A_0 and δA on an arbitrary field $f(\mathbf{x})$ with Fourier transform $f(\mathbf{k})$ is explicitly given by

$$A_0 f(\mathbf{k}) = -\Omega_{\mathbf{k}} f(\mathbf{k}) \quad (86)$$

$$\begin{aligned} \delta Af(\mathbf{k}) = & \mathbf{i}\mathbf{k} \cdot \int_{\mathbf{k}'} \left[\mathbf{i}\mathbf{k}' V(\mathbf{k}') (\psi(\mathbf{k} - \mathbf{k}', t) f(\mathbf{k}')) \right. \\ & \left. + f(\mathbf{k} - \mathbf{k}') \psi(\mathbf{k}') + \frac{\sqrt{2D}}{2\sqrt{\rho_0}} f(\mathbf{k}') \boldsymbol{\xi}(\mathbf{k} - \mathbf{k}', t) \right]. \end{aligned} \quad (87)$$

Using the explicit form of v_0 , we thus find that the Fourier modes of χ evolve according to

$$\begin{aligned} \partial_t \chi(\mathbf{k}, t) = & (\Omega_{\mathbf{q}} - \Omega_{\mathbf{k}}) \chi(\mathbf{k}, t) + \delta A v_0 \\ & + 2v_0(\mathbf{k}) \int_{\mathbf{k}'} v_0(-\mathbf{k}') \Omega_{\mathbf{k}'} \chi(\mathbf{k}', t) - v_0 \int v_0 \delta A v_0. \end{aligned} \quad (88)$$

The symbol $\int_{\mathbf{k}'}$ actually stands for $L^{-3} \sum_{\mathbf{k}'}$. For $\mathbf{k} \neq \pm \mathbf{q}$, the second line of equation (89) vanishes and the evolution equation for χ can be integrated, to yield

$$\chi(\mathbf{k}, t) = \int_{-\infty}^t d\tau e^{-(\Omega_{\mathbf{k}} - \Omega_{\mathbf{q}})(t-\tau)} \delta A v_0(\mathbf{k}, \tau) \quad (89)$$

which converts into the following time-Fourier expression

$$\chi(\mathbf{k}, \omega) = \frac{\delta A v_0(\mathbf{k}, \omega)}{-i\omega + \Omega_{\mathbf{k}} - \Omega_{\mathbf{q}}} \quad (90)$$

where

$$\delta A v_0(\mathbf{k}, \omega) = L^{-3} W^{-1} \sum_{\mathbf{p}=\pm\mathbf{q}} \left[\alpha_{\mathbf{k},\mathbf{p}} \psi(\mathbf{k} - \mathbf{p}, \omega) + \frac{\sqrt{2D}}{2\sqrt{\rho_0}} \mathbf{i}\mathbf{k} \cdot \boldsymbol{\xi}(\mathbf{k} - \mathbf{p}, \omega) \right] \quad (91)$$

and the symbol $\alpha_{\mathbf{k},\mathbf{p}} = \mathbf{i}\mathbf{k} \cdot [\mathbf{i}(\mathbf{k} - \mathbf{p}) V(\mathbf{k} - \mathbf{p}) + \mathbf{i}\mathbf{p} V(\mathbf{p})]$. We will be using the statistical properties of ψ , namely that

$$\langle \psi(\mathbf{k}, \omega) \psi(\mathbf{k}', \omega') \rangle = L^3 \delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \frac{2D\rho_0 \mathbf{k}^2}{\omega^2 + \Omega_{\mathbf{k}}^2} 2\pi \delta(\omega + \omega'), \quad (92)$$

and

$$\langle \psi(\mathbf{k}, \omega) \boldsymbol{\xi}(\mathbf{k}', \omega') \rangle = L^3 \delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \frac{\sqrt{2D\rho_0} \mathbf{i}\mathbf{k}}{-i\omega + \Omega_{\mathbf{k}}} 2\pi \delta(\omega + \omega') \quad (93)$$

to determine those of χ as given by (91) and (92). Equations (93)–(94) stem from the dynamics of ψ

$$\dot{\psi}(\mathbf{k}, t) = -\Omega_{\mathbf{k}} \psi(\mathbf{k}, t) + \sqrt{2D\rho_0} \mathbf{i}\mathbf{k} \cdot \boldsymbol{\xi}. \quad (94)$$

The $\mathbf{k} = \pm \mathbf{q}$ modes of χ can be seen to satisfy the following evolution equation:

$$\partial_t [\chi(\mathbf{q}, t) + \chi(-\mathbf{q}, t)] = \Omega_{\mathbf{q}} (\chi(\mathbf{q}, t) + \chi(-\mathbf{q}, t)) \quad (95)$$

which tells us that $\chi(\pm \mathbf{q}, t)$ is purely imaginary. This is, of course, consistent with the constraint $\int v^2 = 1$ when the latter is expressed to linear order in χ .

The expression of the leading fluctuating correction to the largest Lyapunov exponent is defined by $\delta\lambda$

$$\delta\lambda = \frac{1}{t} \int_0^t dt \int_{\mathbf{x}} [\chi A_0 v_0 + v_0 A_0 \chi + v_0 \delta A v_0]. \quad (96)$$

As will be shown, $\delta\lambda$ renormalizes the fluctuations of λ but not its mean value. The first two integrals in equation (97) yield a contribution proportional to $\chi(\mathbf{q}, t) + \chi(-\mathbf{q}, t)$ and thus vanish. The remaining integral can be expressed in terms of the ψ and ξ fields, as

$$t\delta\lambda = L^{-3} \int dt \left(\frac{\alpha_{-\mathbf{q}, \mathbf{q}}}{2} (\psi(2\mathbf{q}, t) + \psi(-2\mathbf{q}, t)) + \frac{\sqrt{2D}}{4\sqrt{\rho_0}} i\mathbf{q} \cdot (\xi(2\mathbf{q}, t) - \xi(-2\mathbf{q}, t)) \right). \quad (97)$$

Being linear in the fields, the distribution of $\delta\lambda$ is of course Gaussian (although it does not have much meaning in any case to talk about high order cumulants given the Gaussian nature of the RPA). After tedious but standard manipulations, its variance is found to be

$$t\langle\delta\lambda^2\rangle = L^{-3} \frac{D\mathbf{q}^2}{4\rho_0} \left[1 + \frac{4\alpha_{-\mathbf{q}, \mathbf{q}}\rho_0}{\Omega_{2\mathbf{q}}} \right]^2. \quad (98)$$

The coefficient $\alpha_{-\mathbf{q}, \mathbf{q}}$ has the expression $\alpha_{-\mathbf{q}, \mathbf{q}} = q^2(V(\mathbf{q}) - 2V(2\mathbf{q}))$, so that within the RPA approximation,

$$t\langle\delta\lambda^2\rangle = L^{-3} \frac{D\mathbf{q}^2}{4\rho_0} \left[1 + \frac{\beta\rho_0(V(\mathbf{q}) - 2V(2\mathbf{q}))}{1 + \beta\rho_0 V(2\mathbf{q})} \right]^2. \quad (99)$$

The result appearing in (100) is interesting in various respects. First of all, we have not used any fluctuating hydrodynamics, but our expression extrapolated to non-interacting particles coincides with the exact result obtained in [10]. Second, it shows the connection between the Lyapunov exponent and the microscopic interaction between particles, with higher multiples of the slowest mode \mathbf{q} entering higher order fluctuations of λ . This too, though in a different fashion, was seen in [10]. Using our harmonic sphere potential (84), which is mildly repulsive, we find fluctuations of chaoticity to be reduced with respect to those of an ideal gas (because $V(\mathbf{q}) - 2V(2\mathbf{q}) < 0$, a property that other very short range potentials share, such as a screened Coulomb potential for instance). Again, a similar trend can be found in low density lattice gases, but there fluctuations shoot up as the density exceeds a threshold value. It would be dangerous to try to extrapolate the RPA result into a dense regime.

6. Conclusion

In this article, we have presented two different approaches to compute the Lyapunov exponent of stochastic systems, going beyond the standard ‘same-noise’ versus

‘different-noise’ paradigm usually referred to [1–3]. Indeed, we show that enforcing the same noise realization in two stochastic processes is ambiguous, something which is well-known in the damage spreading community [16, 17] but had not been pointed out in the much simpler context of particles diffusing in a fluid. Even without taking into account the full consequences of hydrodynamics, the fact that the noise on a colloidal particle comes from collisions with fluid particles requires a new prescription when comparing two initial conditions experiencing ‘the same noise’.

We have shown that the Lyapunov exponent computed using this environment-based noise has different distributions than the ones stemming from the standard particle-based noise. When the correlation length of the environment is much shorter than the interparticle distance, however, the two approaches become equivalent (as they should).

When considering collective modes, like the density field, we have shown that linearizing the fluctuating hydrodynamics is equivalent to linearizing the microscopic dynamics and then coarse-graining the tangent dynamics. In section 5, we have studied the case of interacting particles, providing a mean-field estimate for the largest Lyapunov exponent.

Our article thus both provides a starting point for future studies of fluctuations of Lyapunov exponents in large interacting stochastic systems and highlights the underlying hypothesis made on the origin of noise in stochastic processes and their importance when dealing with chaos.

Appendix A. Time discretization for the environment-based noise

When time-discretizing the stochastic differential equation (7) during a small time interval Δt , one needs to specify at which time, $t + \varepsilon\Delta t \in [t, t + \Delta t]$, the prefactor of the noise is evaluated:

$$\begin{aligned} r(t + \Delta t) &= r(t) + \int_t^{t+\Delta t} \int dy \delta(y - r(t')) \chi(y, t') dt' \\ &\simeq r(t) + \int dy \delta(y - r(t + \varepsilon\Delta t)) \int_t^{t+\Delta t} \chi(y, t') dt'. \end{aligned} \quad (\text{A.1})$$

As we now show, the Fokker–Planck equation corresponding to (7) is actually independent of ε . Indeed, (A.1) amounts to

$$\Delta r \simeq \int_t^{t+\Delta t} dt' \int dy \delta(y - r(t) - \varepsilon\Delta r) \chi(y, t') \quad (\text{A.2})$$

which is a self-consistent equation on Δr . As $\Delta r \ll 1$, we can perform a series expansion of the Dirac distribution in the integral

$$\Delta r \simeq \int_t^{t+\Delta t} dt' \int dy [\delta(y - r(t)) \chi(y, t') - \varepsilon\Delta r \chi(y, t') \partial_y \delta(y - r(t)) + \dots] \quad (\text{A.3})$$

Replacing Δr in the integral by its expression (A.3) then yields, when $\Delta t \rightarrow 0$,

$$\frac{\langle \Delta r \rangle}{\Delta t} \simeq -\frac{\varepsilon}{\Delta t} \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \int dy \int dy' \partial_y \delta(y - r(t)) \langle \chi(y, t') \chi(y', t'') \rangle. \quad (\text{A.4})$$

Using $\langle \chi(y, t') \chi(y', t'') \rangle = C(y - y') \delta(t' - t'')$ and integrating over t' and t'' then yields

$$\begin{aligned} \frac{\langle \Delta r \rangle}{\Delta t} &= 2D\varepsilon \int dy \int dy' \delta(y - r(t)) \delta(y' - r(t)) C'(y - y') + \mathcal{O}_{\Delta t \rightarrow 0}(\Delta t) \\ &= 2D\varepsilon C'(0) + \mathcal{O}_{\Delta t \rightarrow 0}(\Delta t). \end{aligned} \quad (\text{A.5})$$

Finally, when Δt tends to 0, the limit of the moment rate is

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta r \rangle}{\Delta t} = 2D\varepsilon C'(0). \quad (\text{A.6})$$

Since C is even, $C'(0) = 0$, and the first moment rate, which is also the first coefficient of the Kramers–Moyal expansion, is zero for any choice of time discretization. All the discretizations $\varepsilon \in [0, 1]$ are then equivalent.

Appendix B. Stratonovitch calculus

The previous Langevin equation is independent of the discretization. In this section, we will consider this equation as a Stratonovitch equation.

The local density is defined by

$$\rho(x, t) = \sum_{j=1}^N \delta(x - r_j(t)). \quad (\text{B.1})$$

In order to find the Langevin equation the density evolves according to, we perform a Kramers–Moyal expansion of the moments $\Delta \rho = \rho(t + \Delta t) - \rho(t)$, averaged over the realization of the noise during the time interval $[t, t + \Delta t]$, at a fixed value of $\rho(t)$. We use the Stratonovitch discretization that allows us to manipulate singular functions within the framework of standard differential calculus, so that

$$\partial_t \rho = - \sum_{j=1}^N \int dy \chi(y, t) \delta(y - r_j) \partial_x \delta(x - r_j) \quad (\text{B.2})$$

which actually means that:

$$\Delta \rho = - \sum_{j=1}^N \int dy \delta(y - r_j(t) - \frac{\Delta r_j}{2}) \partial_x \delta(x - r_j(t) - \frac{\Delta r_j}{2}) \int_t^{t+\Delta t} d\tau \chi(y, \tau).$$

Once expanded to leading order in Δt , given that

$$\Delta r_j = \int dy \delta(y - r_j - \frac{\Delta r_j}{2}) \int_t^{t+\Delta t} d\tau \chi(y, \tau), \quad (\text{B.3})$$

we arrive at

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \rho \rangle}{\Delta t} = D \sum_{j=1}^N \int dy dy' [\delta(y - r_j) \partial_x^2 \delta(x - r_j) + \partial_y \delta(y - r_j) \partial_x \delta(x - r_j)] C(y - y')$$

which leads to

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \rho \rangle}{\Delta t} = D \partial_x^2 \rho \quad (\text{B.4})$$

since $C(0) = 1$ and $C'(0) = 0$. Similarly, we find that

$$\lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \rho(x, t) \Delta \rho(x', t') \rangle}{\Delta t} = \sum_{i,j=1}^N \int dy dy' \delta(y - r_i) \delta(y' - r_j) \partial_x \delta(x - r_i) \partial_{x'} \delta(x' - r_j) C(y - y')$$

which gives us

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \rho(x, t) \Delta \rho(x', t') \rangle}{\Delta t} &= 2D \sum_{i,j=1}^N \partial_x \partial_{x'} (\delta(x - r_i) \delta(x' - r_j) C(r_i - r_j)) \\ &= 2D \partial_x \partial_{x'} (\rho(x, t) \rho(x', t) C(x - x')). \end{aligned} \quad (\text{B.5})$$

We can now express the Langevin equation (in Ito's discretization) for the density:

$$\partial_t \rho = D \partial_x^2 \rho - \partial_x \Gamma(x, t) \quad (\text{B.6})$$

with

$$\langle \Gamma(x, t) \rangle = 0 \quad (\text{B.7})$$

$$\langle \Gamma(x, t) \Gamma(x', t') \rangle = 2D \rho(x, t) \rho(x', t) C(x - x') \quad (\text{B.8})$$

Appendix C. Tangent dynamics of particle-based noise: hydrodynamics derivation versus linearization

In this appendix we show that the noise

$$\xi_u^D(x, t) = \left[\frac{u(x, t)}{2\sqrt{\rho(x, t)}} \eta(x, t) \right] \quad (\text{C.1})$$

obtained by linearizing the fluctuating hydrodynamics

$$\dot{\rho}(x, t) = \partial_x^2 \rho(x, t) - \partial_x [\sqrt{\rho(x, t)} \eta(x, t)] \quad (\text{C.2})$$

is equivalent to the one obtained by coarse-graining the microscopic tangent dynamics:

$$\xi_u(x, t) = \partial_x \sum_{j=1}^N \delta r_j \delta(x - r_j(t)) \eta_j(t). \quad (\text{C.3})$$

Since these noises are Gaussian, we simply have to show that their mean and variance are equal. Both means are trivially zero, and we thus only consider their variances.

In section 3.1, we showed that the noise

$$\xi(x, t, \{r_j\}) = \sum_{j=1}^N \rho_j(x, t) \eta_j(t) = \sum_{j=1}^N \delta(x - r_j(t)) \eta_j(t) \quad (\text{C.4})$$

is equivalent to

$$\xi^D(x, t, \{r_j\}) = \partial_x [\sqrt{\rho(x, t)} \eta(x, t)] = \left[\left(\sum_{j=1}^N \delta(x - r_j(t)) \right)^{1/2} \eta(x, t) \right] \quad (\text{C.5})$$

where we have explicitly written the dependence of ξ and ξ^D on r_j .

We will now linearize ξ and ξ^D with respect to the r_j 's and show that the corresponding linearized noises correspond to ξ_u^D and ξ_u . Since $\xi = \xi^D$, this will establish the equivalence between ξ_u and ξ_u^D .

We first look at

$$\mathcal{C}(x, x', t, t') \equiv \langle [\xi(x, t, \{r_j + \delta r_j\}) - \xi(x, t, \{r_j\})][\xi(x', t', \{r_j + \delta r_j\}) - \xi(x', t', \{r_j\})] \rangle.$$

Using the explicit expression of ξ , one gets

$$\begin{aligned} \frac{\mathcal{C}(x, x', t, t')}{2D} &= \sum_{i,j=1}^N [\delta(x - r_i(t) - \delta r_i) - \delta(x - r_i(t))] \\ &\quad \times [\delta(x' - r_j(t') - \delta r_j) - \delta(x' - r_j(t'))] \frac{\langle \eta_i(t) \eta_j(t') \rangle}{2D} \\ &= \sum_{i,j=1}^N [\delta(x - r_i(t) - \delta r_i) - \delta(x - r_i(t))] \\ &\quad \times [\delta(x' - r_j(t) - \delta r_j) - \delta(x' - r_j(t))] \delta_{i,j} \delta(t - t') \\ &= \delta(t - t') \sum_{i=1}^N [\delta(x - r_i - \delta r_i) - \delta(x - r_i)] [\delta(x' - r_i - \delta r_i) - \delta(x' - r_i)] \\ &\simeq \delta(t - t') \sum_{i=1}^N [\delta r_i^2 \partial_x \delta(x - r_i) \partial_{x'} \delta(x' - r_i) + \mathcal{O}(\delta r_i^3)]. \end{aligned} \quad (\text{C.6})$$

The leading order in δr of $\mathcal{C}(x, x', t, t')$ is exactly the correlation of $\xi_u(x, t)$.

Let us now calculate the same quantity for $\xi^D(x, t)$:

$$\begin{aligned}
\frac{\mathcal{C}^D(x, x', t, t')}{2D} &= \left(\left[\sum_{i=1}^N \delta(x - r_i(t) - \delta r_i) \right]^{1/2} - \left[\sum_{i=1}^N \delta(x - r_i(t)) \right]^{1/2} \right) \\
&\times \left(\left[\sum_{i=1}^N \delta(x' - r_i(t') - \delta r_i) \right]^{1/2} - \left[\sum_{i=1}^N \delta(x' - r_i(t')) \right]^{1/2} \right) \\
&\times \frac{\langle \eta(x, t) \eta(x', t') \rangle}{2D} \\
&= \delta(x - x') \delta(t - t') \left[\underbrace{\sum_{i=1}^N \delta(x - r_i - \delta r_i) + \sum_{i=1}^N \delta(x - r_i)}_{\text{I}} \right. \\
&\quad \left. - 2 \underbrace{\left(\sum_{i,j=1}^N \delta(x - r_i - \delta r_i) \delta(x - r_j) \right)^{1/2}}_{\text{II}} \right].
\end{aligned}$$

We now perform a series expansion to second order in δr_j . The term **I** gives

$$\sum_{i=1}^N \left[\delta(x - r_i) - \delta r_i \partial_x \delta(x - r_i) + \frac{1}{2} \delta r_i^2 \partial_x^2 \delta(x - r_i) \right] \quad (\text{C.7})$$

whereas the term **II** yields

$$2\rho - \sum_{i=1}^N \delta r_i \partial_x \delta(x - r_i) + \frac{1}{2} \sum_{i=1}^N \delta r_i^2 \partial_x^2 \delta(x - r_i) + \frac{1}{4\rho} \left[\sum_{i=1}^N \delta r_i \partial_x \delta(x - r_i) \right]^2.$$

Noting that $\rho = \sum_{i=1}^N \delta(x - r_i)$, the overall expression simplifies and one gets

$$\begin{aligned}
\frac{\mathcal{C}^D(x, x', t, t')}{2D} &\simeq \frac{1}{4\rho(x, t)} \left[\sum_{i=1}^N \delta r_i \partial_x \delta(x - r_i) \right]^2 \delta(x - x') \delta(t - t') + \mathcal{O}(\delta r^3) \\
&\simeq \frac{u(x, t)^2}{4\rho(x, t)} \delta(x - x') \delta(t - t') + \mathcal{O}(\delta r^3) \\
&\simeq \frac{u(x, t)}{2\sqrt{\rho(x, t)}} \frac{u(x', t')}{2\sqrt{\rho(x', t')}} \delta(x - x') \delta(t - t') + \mathcal{O}(\delta r^3) \quad (\text{C.8})
\end{aligned}$$

which is, at the leading order in δr , the correlation of $\xi_u^D(x, t)$. Since $\xi(x, t) = \xi^D(x, t)$, we have $\mathcal{C}(x, x', t, t') = \mathcal{C}^D(x, x', t, t')$, which implies that $\xi_u(x, t)$ and $\xi_u^D(x, t)$ have the same correlations, and are thus equivalent.

References

- [1] Benzi R, Paladin G, Parisi G and Vulpiani A 1985 *J. Phys. A: Math. Theor.* **18** 2157
- [2] Arnold L, Kliemann W and Oeljeklaus E 1986 Lyapunov exponents of linear stochastic systems *Lyapunov Exponents (Lecture Notes in Mathematics vol 1186)* ed L Arnold and V Wihstutz (Berlin: Springer) pp 85–125
- [3] Arnold L 1988 Lyapunov exponents of nonlinear stochastic systems *Nonlinear Stochastic Dynamic Engineering Systems: IUTAM Symposium* ed F Ziegler and G Schuëller (Berlin: Springer) pp 181–201
- [4] Grassberger P, Badii R and Politi A 1988 *J. Stat. Phys.* **51** 135–78
- [5] Graham R 1988 *Europhys. Lett.* **5** 101
- [6] Dean D 1996 *J. Phys. A: Math. Gen.* **29** L613
- [7] Nardini C, Gupta S, Ruffo S, Dauxois T and Bouchet F 2012 *J. Stat. Mech.* L01002
- [8] Nardini C, Gupta S, Ruffo S, Dauxois T and Bouchet F 2012 *J. Stat. Mech.* P12010
- [9] Bouchet F, Nardini C and Tangarife T 2013 *J. Stat. Phys.* **153** 572–625
- [10] Laffargue T, Sollich P, Tailleur J and van Wijland F 2015 *Europhys. Lett.* **110** 10006
- [11] Kim B, Kawasaki K, Jacquin H and van Wijland F 2014 *Phys. Rev. E* **89** 012150
- [12] Solon A, Cates M and Tailleur J 2015 arXiv:1504.07391
- [13] Kipnis C, Marchioro C and Presutti E 1982 *J. Stat. Phys.* **27** 65–74
- [14] Dean D S and Podgornik R 2014 *Phys. Rev. E* **89** 032117
- [15] Démercy V, Bénichou O and Jacquin H 2014 *New J. Phys.* **16** 053032
- [16] Grassberger P 1995 *J. Stat. Phys.* **79** 13–23
- [17] Hinrichsen H, Weitz J S and Domany E 1997 *J. Stat. Phys.* **88** 617–36