Fermionic field theory for directed percolation in (1 + 1)-dimensions

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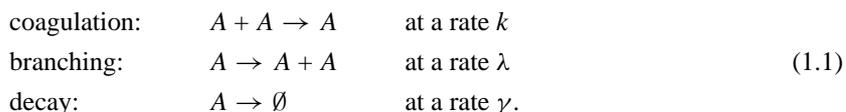§ Laboratoire de Physique Théorique‖, Université de Paris-Sud, 91405 Orsay Cedex, France.

Abstract. We formulate directed percolation in $(1 + 1)$ dimensions in the language of a reaction–diffusion process with exclusion taking place in one space dimension. We map the master equation that describes the dynamics of the system onto a quantum spin chain problem. From there we build an interacting fermionic field theory of a new type. We study the resulting theory using renormalization group techniques. This yields numerical estimates for the critical exponents and provides a new alternative analytic systematic procedure to study low-dimensional directed percolation.

1. Introduction

1.1. Microscopic model

Each site of a one-dimensional lattice $(\mathbb{Z})$ is initially occupied by one particle A with probability ρ or empty with probability $1 - \rho$. The A particles perform simple (continuous time) random walk with a diffusion constant D. We further impose the exclusion constraint, namely, each site is occupied by at most one particle. In addition the particles may undergo several reaction processes:

$$
\begin{aligned}
\text{coagulation:} &\quad A + A \rightarrow A &\quad \text{at a rate } k \\
\text{branching:} &\quad A \rightarrow A + A &\quad \text{at a rate } \lambda \\
\text{decay:} &\quad A \rightarrow \emptyset &\quad \text{at a rate } \gamma.
\end{aligned}
\tag{1.1}
$$

Owing to the exclusion constraint particles react when they sit on neighbouring sites. Similarly, diffusion takes place only when empty sites allow it to. Coagulation, branching and decay, along with diffusive motion, define a reaction–diffusion process that has already received considerable attention in the past: this is the Schlögl autocatalytic reaction, which is known to belong to the universality class of directed percolation. The d-dimensional generalization of this model has been studied via renormalization group techniques by means of ε-expansion in the vicinity of the upper critical dimension $d_c = 4$ [1]. The very few analytic results that exist in one dimension are based on short-time series expansions [2]. Our aim in this work is to provide a systematic approximation scheme specific to $d = 1$.

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1.2. Mean field for directed percolation

It is possible to write a mean-field equation for the average local particle density $n(t)$ at time t:

$$\frac{dn}{dt} = (\lambda - \gamma)n - (k + \lambda)n^2. \tag{1.2}$$

From this equation one predicts that in the steady state

$$n(\infty) = \begin{cases} \dfrac{\lambda - \gamma}{k + \lambda} & \text{if} \quad \lambda > \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Hence mean field predicts a continuous transition between an active state for $\gamma < \lambda$ in which a finite fraction of A survives indefinitely, and an absorbing state in which A have completely disappeared forever, which occurs for $\gamma \geqslant \lambda$. In the following we shall use γ as the control parameter and fix all other parameters. At the mean-field level we see that the steady state of this system undergoes a second-order phase transition between an active state with nonzero A density, at $\gamma < \lambda$, and an absorbing A-free state at $\gamma > \lambda$. Within the mean-field picture the transition occurs at the critical value $\gamma_c = \lambda$. It is possible to summarize the scaling properties of the particle density in a single formula

$$n(t) = b^{-\frac{1+\eta}{2}} \mathcal{F}(b^{-z}t, b^{1/\nu}|\gamma - \gamma_c|) \tag{1.4}$$

which holds for $b \gg 1$ with the arguments of $\mathcal{F}$ fixed. This scaling relation defines the critical exponents η, z and ν. Their mean-field values are $\eta = 0$, $z = 2$ and $\nu = \frac{1}{2}$. In the steady state the density behaves as $|\gamma - \gamma_c|^\beta$ as $\gamma \to \gamma_c^-$, which defines the exponent $\beta = \nu(1 + \eta)/2$.

1.3. Motivations and outline

From the analytic point of view there are very few exact or even approximate results on directed percolation in low space dimension (see [3, 4] for recent reviews). Owing to its ubiquitous nature in the study of stochastic processes, directed percolation has become the paradigm of out-of-equilibrium systems possessing a second-order phase transition in their steady state. Our aim is to remedy the scarcity of analytic techniques specific to low and physically relevant space dimensions. Indeed $d = 1$ is the relevant dimension for instance in the study of surface growth phenomena. Another motivation comes from particle physics. There the branching–coagulation language is used as a phenomenological description of hadronic high-energy scattering processes with $d = 2$ being the physical dimension corresponding to the number of transverse space directions. Other applications include the study of intermittency (see e.g. Henkel and Peschanski [5]), either in the Schwinger mechanism [6], or in turbulence [7]. Fluctuations play an increasing role as the dimension is decreased below the upper critical dimension $d_c = 4$, which provides further motivation to focus on low dimensions. In the absence of any exact solution we believe that our method provides new insight into the peculiarities of one-dimensional directed percolation.

The outline of this paper is as follows. In section 2 we map the master equation that describes the reaction–diffusion process equation (1.1) first onto a spin chain problem. The spin chain is then mapped onto a fermionic field theory. The procedure, which we describe in great detail, consists in building a fermionic field theory starting from a non-Hermitian Hamiltonian originating from the stochastic process equation (1.1). This raises a number of difficulties which, to our knowledge, appear for the first time in the literature. These are exact mappings. We then proceed with the analysis of the full theory describing directed percolation using renormalization group techniques. Our calculations yield numerical estimates for the critical exponents.

2. From the master equation to a quantum spin chain

2.1. Master equation

A microstate of the system described in section (1.1) is characterized by the set of occupation numbers $\{n_j\}_{j \in \mathbb{Z}}$ defined by

$$
n_j =
\begin{cases}
1 & \text{if site } j \text{ is occupied by an } A \\
0 & \text{if site } j \text{ is empty.}
\end{cases}
\tag{2.1}
$$

We now define the *spin* variable $s_j \equiv 2n_j - 1$. Let $s \equiv \{s_j\}$ denote a generic microstate of the system and let it index a set of vectors $|s\rangle$ in a Hilbert space. The master equation for the probability of occurence $P(s, t)$ of state s at time t is equivalent to an evolution equation for the linear combination

$$
|\Phi(t)\rangle \equiv \sum_s P(s, t)|s\rangle
\tag{2.2}
$$

which reads

$$
\frac{\mathrm{d}|\Phi(t)\rangle}{\mathrm{d}t} = -\hat{H}|\Phi(t)\rangle
\tag{2.3}
$$

where $\hat{H}$ is an evolution operator (also abusively called a *Hamiltonian*) acting in the Hilbert space spun by the states $|s\rangle$. We introduce the Pauli matrices $\vec{\sigma}_i$ defined by

$$
\sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_i^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\tag{2.4}
$$

and also define the raising and lowering operators

$$
\sigma_i^{\pm} \equiv \tfrac{1}{2}(\sigma_i^x \pm \mathrm{i}\sigma_i^y).
\tag{2.5}
$$

We thus have the identities $\sigma_i^z = 2\sigma_i^+\sigma_i^- - 1$ and $\sigma_i^+\sigma_i^- + \sigma_i^-\sigma_i^+ = 1$. The variable s_i is the eigenvalue of σ_i^z. One may verify that $\hat{H}$ may be written in the form

$$
\hat{H} = \hat{H}_{\text{diffusion}} + \hat{H}_{\text{decay}} + \hat{H}_{\text{branching}} + \hat{H}_{\text{coagulation}}
\tag{2.6}
$$

where we have set

$$
\hat{H}_{\text{diffusion}} = -\frac{D}{2} \sum_i \vec{\sigma}_i . \vec{\sigma}_{i+1}
\tag{2.7}
$$

$$
\hat{H}_{\text{decay}} = \gamma \sum_i (\tfrac{1}{2}\sigma_i^z - \sigma_i^-)
\tag{2.8}
$$

$$
\hat{H}_{\text{branching}} = -\frac{\lambda}{4} \sum_i (\sigma_i^z\sigma_{i+1}^z + 2\sigma_i^+ + \sigma_i^+\sigma_{i+1}^z + \sigma_{i+1}^+\sigma_i^z)
\tag{2.9}
$$

$$
\hat{H}_{\text{coagulation}} = \frac{k}{4} \sum_i (2\sigma_i^z - 2\sigma_i^- + \sigma_i^z\sigma_{i+1}^z - \sigma_i^-\sigma_{i+1}^z - \sigma_{i+1}^-\sigma_i^z).
\tag{2.10}
$$

Note that we have dropped all constant terms in equations (2.7)–(2.10). They ensure the conservation of probability but will however play no role in the subsequent analysis.

2.2. General properties of the spin chain and average of observables

In this section we recall for completeness some of the properties of the spin chain Hamiltonian defined by equation (2.3). We need to introduce a *projection state* $\langle p|$ defined by

$$
\langle p| \equiv \sum_s \langle s|.
\tag{2.11}
$$

Given a physical observable $A(s)$ we denote by $\hat{A}$ the operator obtained by replacing in the explicit expression of A the variable s_i by the operator σ_i^z. For instance the choice $A(s) = \frac{1}{2}(s_j + 1)$, which is the local number of particles at site j, leads to $\hat{A} = \frac{1}{2}(\sigma_j^z + 1)$. The average of the observable $A(s)$ may be expressed as

$$\langle A(s) \rangle(t) = \langle p | \hat{A} | \Phi(t) \rangle \tag{2.12}$$

as was first noted by Felderhof [8].

Conservation of probability imposes that $\langle p |$ is a left eigenstate of $\hat{H}$ with eigenvalue 0:

$$\langle p | \hat{H} = 0 \tag{2.13}$$

from which it follows that

$$\forall t \qquad \langle p | \Phi(t) \rangle = 1. \tag{2.14}$$

Additionally, $\hat{H}$ has at least one right eigenvector with eigenvalue 0, which describes the stationary state of the system. The eigenvalues of $\hat{H}$ all have a positive real part. Other details may be found in the reviews by Alcaraz *et al* [9] or Henkel *et al* [10]. For our purposes we need one more property of the projection state. It is based on the following identity:

$$e^{\sigma_i^-} \sigma_i^+ = (\sigma_i^+ - \sigma_i^- - 2\sigma_i^+ \sigma_i^- + 1)e^{\sigma_i^-}. \tag{2.15}$$

After noting that

$$\langle p | = \langle -1 | e^{\sum_j \sigma_j^-} \tag{2.16}$$

it becomes possible to express the average of an observable $A(s)$ in the form

$$\langle A(s) \rangle(t) = \langle -1 | \tilde{A}' e^{-\tilde{H}t} e^{\sum_j \sigma_j^-} | \Phi(0) \rangle \tag{2.17}$$

where $\tilde{A}'$ and $\tilde{H}$ are deduced from $\hat{A}$ and $\hat{H}$ as follows. Express these operators only in terms of the σ^+, σ^- and $\sigma^+\sigma^-$. For each j replace in the resulting expression σ_j^+ by $\sigma_j^+ - \sigma_j^- - 2\sigma_j^+\sigma_j^- + 1$. This yields the operators $\tilde{A}$ and $\tilde{H}$. In the expression of $\tilde{A}$ one first puts all the σ^+ to the left of the σ^- then one formally sets the σ^+ to 0. This yields the operator $\tilde{A}'$. With the particular expression of $\hat{H}$ equation (2.6) one finds after straightforward manipulations:

$$\tilde{H} = -\left(D + \frac{\lambda}{2}\right) \sum_i [\sigma_i^- \sigma_{i+1}^+ + \sigma_i^+ \sigma_{i+1}^- - 2\hat{n}_i] + (\gamma - \lambda) \sum_i \hat{n}_i + (\lambda + k - 2D) \sum_i \hat{n}_i \hat{n}_{i+1}$$

$$+ \frac{1}{2}(\lambda + k) \sum_i [\sigma_i^- \hat{n}_{i+1} + \hat{n}_i \sigma_{i+1}^-] - \frac{\lambda}{2} \sum_i [\sigma_i^+ \hat{n}_{i+1} + \hat{n}_i \sigma_{i+1}^+] \tag{2.18}$$

where we have adopted the notation $\hat{n}_i \equiv \sigma_i^+ \sigma_i^-$. As we have already mentioned we have omitted a constant term in $\tilde{H}$. In order to find this constant, one would have to push the σ^+ to the left of the σ^- then formally set to 0 all the σ^+, which produces an operator $\tilde{H}'$ and adjust the constant so that $\langle -1 | \tilde{H}' = 0$.

3. Fermionic field theory

3.1. Jordan–Wigner transformation

We define a set of fermionic creation and annihilation operators $c_i^\dagger$ and c_i by

$$\sigma_i^+ = c_i^\dagger \exp\left[i\pi \sum_{j<i} \hat{n}_j\right] \qquad \sigma_i^- = \exp\left[-i\pi \sum_{j<i} \hat{n}_j\right] c_i \tag{3.1}$$

where $\hat{n}_j = \sigma_i^+ \sigma_i^- = c_j^\dagger c_j$ is the particle number operator. It has the eigenvalue n_j. This is the Jordan–Wigner transformation. The commutation relations between the c_i and $c_j^\dagger$ are

$$\{c_i, c_j^\dagger\} = \delta_{ij} \qquad \{c_i^\dagger, c_j^\dagger\} = 0 = \{c_i, c_j\}. \tag{3.2}$$

An equivalent expression for the *string* operator in equation (3.1) is

$$e^{\pm i\pi \sum_{\ell<i} \hat{n}_\ell} = \prod_{\ell<i}(1 - 2\hat{n}_\ell). \tag{3.3}$$

As an example, for diffusion alone, the evolution operator would read

$$\tilde{H}_{\text{diffusion}} = -D \sum_i [c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i - 2\hat{n}_i + 2\hat{n}_i \hat{n}_{i+1}]. \tag{3.4}$$

It is surprisingly left invariant by the transformation equation (2.15). In the quadratic part of equation (3.4) one may recognize the diffusion process, that results in a Laplacian. However, diffusion takes place with the constraint of excluded volume, which accounts for the quartic interaction term.

3.2. Coherent states

As we have seen in the previous section, the average of a physical observable can be cast in the form

$$\langle A \rangle(t) = \langle -1|\tilde{A}' e^{-\tilde{H}t} e^{\sum_i \sigma_i^-}|\Phi(0)\rangle. \tag{3.5}$$

We shall restrict our analysis to the case of an initial state with particles randomly and independently distributed with local density ρ_j at site j, that is with the initial distribution

$$P(n, t = 0) = \prod_j [\rho_j \delta_{n_j,1} + (1 - \rho_j)\delta_{n_j,0}]. \tag{3.6}$$

Under those assumptions it is easy to check that

$$|\Phi'(0)\rangle \equiv e^{\sum_i \sigma_i^-}|\Phi(0)\rangle = e^{\sum_j \rho_j \sigma_j^+}| -1\rangle. \tag{3.7}$$

At this stage we will denote by $|0\rangle$ the vacuum (i.e. particle-free) state, instead of $| -1\rangle$, as the former notation is more appropriate in the particle number language. To summarize, the average of an observable A reads

$$\langle A \rangle(t) = \langle 0|\tilde{A}' e^{-\tilde{H}t}|\Phi'(0)\rangle. \tag{3.8}$$

We rewrite the exponential factor in equation (3.8) using the Trotter formula:

$$e^{-\tilde{H}t} = \lim_{N \to \infty}\left(1 - \frac{\tilde{H}t}{N}\right)^N. \tag{3.9}$$

We now introduce the coherent states associated to the creation and annihilation operators. Let $|\psi_{ij}\rangle$, for $j = 0, \ldots, N$ and $i \in \mathbb{Z}$ denote a set of coherent states associated to c_i and $c_i^\dagger$:

$$|\psi_{ij}\rangle = e^{-\frac{1}{2}\psi_{ij}^* \psi_{ij}} e^{-\psi_{ij} c_i^\dagger}|0\rangle$$
$$\langle \psi_{ij}| = \langle 0|e^{-\frac{1}{2}\psi_{ij}^* \psi_{ij}} e^{-c_i \psi_{ij}^*}. \tag{3.10}$$

These coherent states are indexed by a pair of conjugate Grassmann variables ψ_{ij} and ψ_{ij}^*. By definition they have the property that

$$c_i|\psi_{ij}\rangle = \psi_{ij}|\psi_{ij}\rangle \qquad \langle \psi_{ij}|c_i^\dagger = \langle \psi_{ij}|\psi_{ij}^*. \tag{3.11}$$

With the definition equation (3.10) one may verify that

$$\langle \psi_{ij} | \psi_{ij} \rangle = 1 \tag{3.12}$$

and that, with the integration convention $\int d\psi \, \psi = \int d\psi^* \psi^* = 1$, one has

$$\int d\psi_{ij}^* \, d\psi_{ij} e^{-\psi_{ij}^* \psi_{ij}} \psi_{ij}^n \psi_{ij}^{*m} = \delta_{nm} \tag{3.13}$$

which implies

$$\int d\psi_{ij}^* \, d\psi_{ij} | \psi_{ij} \rangle \langle \psi_{ij} | = \mathbf{1}. \tag{3.14}$$

Equation (3.14) generalized to the state

$$|\psi_j\rangle \equiv \otimes_i |\psi_{ij}\rangle \tag{3.15}$$

leads us to

$$e^{-\tilde{H}t} = \int \prod_{j=0}^{N} \prod_i [d\psi_{ij}^* \, d\psi_{ij}] |\psi_{j=N}\rangle \langle \psi_N | 1 - \tilde{H} \frac{t}{N} |\psi_{N-1}\rangle \cdots \langle \psi_1 | 1 - \tilde{H} \frac{t}{N} |\psi_0\rangle \langle \psi_0 |. \tag{3.16}$$

Thus we need the quantities

$$H_{j,j-1} \equiv \frac{\langle \psi_j | \tilde{H} | \psi_{j-1} \rangle}{\langle \psi_j | \psi_{j-1} \rangle} \qquad j = 1, \ldots, N \tag{3.17}$$

as well as

$$\langle \psi_j | \psi_{j-1} \rangle = e^{-\sum_i \psi_{ij}^* (\psi_{ij} - \psi_{i,j-1})} e^{\frac{1}{2} \sum_i \psi_{ij}^* \psi_{ij} - \frac{1}{2} \sum_i \psi_{i,j-1}^* \psi_{i,j-1}}. \tag{3.18}$$

In order to obtain $H_{j,j-1}$ one replaces in the normal ordered form of $\tilde{H}$ the annihilators c_ℓ by $\psi_{\ell,j-1}$ and the creators $c_m^\dagger$ by ψ_{mj}^*. The projection onto the initial state $|\Phi'(0)\rangle$ reads

$$\langle \psi_{j=0} | \Phi'(0) \rangle = \ldots e^{\rho_i \psi_{i0}^*} e^{\rho_{i+1} \psi_{i+1,0}^*} \ldots \equiv P_i[\psi^*(x, 0)]. \tag{3.19}$$

In a similar way, one finds

$$\langle 0 | \tilde{A}' | \psi_{j=N} \rangle = A'(\psi_N) \tag{3.20}$$

where A' is some function of ψ_N the precise determination of which we postpone to section 3.5.

3.3. Continuous field theory

It is convenient to split $H_{j+1,j}$ in the form

$$H_{j+1,j} = \mathsf{E}_{j+1,j} + \mathsf{O}_{j+1,j} \tag{3.21}$$

where $\mathsf{E}_{j+1,j}$ and $\mathsf{O}_{j+1,j}$ denote the even and odd components of $H_{j+1,j}$, respectively. In the limit $\Delta t \to 0$ the ordered product

$$\overleftarrow{\Pi} \equiv (1 - \Delta t \, H_{N,N-1}) \cdots (1 - \Delta t \, H_{1,0}) \tag{3.22}$$

does not reduce to a simple exponential as the terms in the product do not commute. We may use an identity [11] for a sequence of odd Grassmann variables $\alpha_1, \ldots, \alpha_N$:

$$e^{\alpha_N} \ldots e^{\alpha_1} = e^{\sum_n \alpha_n + \sum_{n=1}^{N} \sum_{m=1}^{n-1} \alpha_n \alpha_m}. \tag{3.23}$$

Hence we may rewrite $\overleftarrow{\Pi}$ in the form

$$\overleftarrow{\Pi} = \exp\left[-\Delta t \sum_j H_{j+1,j} + \Delta t^2 \sum_{j=0}^{N-1} \sum_{\ell=0}^{j-1} \mathsf{O}_{j+1,j} \mathsf{O}_{\ell+1,\ell} \right] \tag{3.24}$$

which yields in the continuum limit:

$$\overleftarrow{\Pi} = \exp\Bigg[-\int_0^{t_f} \mathrm{d}t\, H[\psi(t), \psi^*(t)]$$
$$+ \int_0^{t_f} \mathrm{d}t\, \mathrm{d}t'\, \Theta(t - t')\mathsf{O}[\psi(t), \psi^*(t)]\mathsf{O}[\psi(t'), \psi^*(t')] \Bigg]. \tag{3.25}$$

We define an action S by

$$S[\bar\psi, \psi] = \int \mathrm{d}x\, \mathrm{d}t\, \bar\psi \partial_t \psi + \int \mathrm{d}t\, H[\psi, \bar\psi]$$
$$- \int \mathrm{d}t\, \mathrm{d}t'\, \Theta(t - t')\mathsf{O}[\psi(t), \bar\psi(t)]\mathsf{O}[\psi(t'), \bar\psi(t')] \tag{3.26}$$

where the notation $H[\psi(t), \bar\psi(t)]$ means the (normal-ordered) $\tilde H$ in which the creators and annihilators have been replaced by their Grassmann eigenvalues after the former have been moved to the left of the latter. The quantity $\mathsf{O}[\psi(t), \bar\psi(t)]$ is the odd component of $H[\psi(t), \bar\psi(t)]$. Thus we come up with an action that comprises terms that are nonlocal in space and time. In what follows it could be possible to proceed entirely within the fermion operator formalism (pertubation expansions have been checked this way). There one would have to use time-ordered products in evaluating correlation functions, the effect of which is precisely to rebuild effectively nonlocal interactions.

Again using equation (3.23), we find

$$P_i[\bar\psi(x, 0)] = \exp\Bigg[\sum_i \rho_i \bar\psi_i(0) + \sum_{i<j} \rho_i \rho_j \bar\psi_i(0)\bar\psi_j(0) \Bigg] \tag{3.27}$$

which allows us to express the average of the physical observable A in a path integral form as follows:

$$\langle A\rangle(t_f) = \int \mathcal{D}\bar\psi \mathcal{D}\psi\, A'[\psi(x, t_f)]\mathrm{e}^{-S[\bar\psi, \psi]} P_i[\bar\psi(x, 0)]. \tag{3.28}$$

We are now in a position to apply the existing techniques of field theory.

3.4. Comparison with bosonic field theories for stochastic processes

To conclude this section we would like to make a comparison with reaction–diffusion problems in which the exclusion constraint is not imposed [12, 13]. There bosonic operators $(a_i, a_i^\dagger)$, rather than spin operators, are used to express the evolution operator $\hat H$. The projection state reads

$$\langle p| = \langle 0|\mathrm{e}^{\sum_i a_i} \tag{3.29}$$

and the bosonic counterpart to identity equation (2.15) is $\mathrm{e}^{a_i} a_i^{\dagger n} = (a_i^\dagger + 1)^n \mathrm{e}^{a_i}$. The commutation of the factor $\mathrm{e}^{\sum_i a_i}$ through the $\mathrm{e}^{-\hat H t}$ can certainly be performed from the outset, but is not compulsory. It is possible by an appropriate shift of the field associated to the creation operators to perform this transformation directly within the functional integral formulation. In the present case it seems that such an *a posteriori* nonlinear change of Grassmann field having the same effect as the nontrivial transformation equation (2.15) would be difficult to find and to handle.

3.5. Expectation values of some observables

The main observable of interest is the local particle number $A = n_i$. Hence $\hat{A} = \hat{n}_i$ and $\tilde{A} = \hat{n}_i + \sigma_i^-$, so that $\tilde{A}' = \sigma_i^-$. Since

$$\langle 0|\tilde{A}' = \langle 0|c_i \tag{3.30}$$

we can write the average of the local particle number at site i as the functional integral

$$\langle n_i \rangle(t) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi_i(t)\mathrm{e}^{-S[\psi,\bar{\psi}]}P_i[\bar{\psi}]. \tag{3.31}$$

Another physical quantity of interest is the equal time two-point correlation function $A = n_i n_j$, with $i < j$. The associated $\tilde{A}'$ reads

$$\tilde{A}' = \sigma_i^-\sigma_j^- = \prod_{\ell=i}^{j-1}(1 - 2\hat{n}_\ell)c_j c_i. \tag{3.32}$$

Hence $\langle 0|\tilde{A}' = \langle 0|c_j c_i$ which yields the equal time correlation function

$$\langle n_i n_j \rangle(t) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \psi_j(t)\psi_i(t)\mathrm{e}^{-S[\psi,\bar{\psi}]}P_i. \tag{3.33}$$

For $i = j$ the above formula does not hold, and one finds $\tilde{A}' = c_i$ so that one recovers $\langle n_i^2 \rangle = \langle n_i \rangle$. Other equal time averages can be derived in a similar fashion.

4. Directed percolation

4.1. Action

We now return to our problem of interest, directed percolation. It is described by the evolution operator equation (2.18), which splits into several contributions. We find it convenient to introduce the nonlocal functional

$$\xi_i(t) \equiv \prod_{\ell \leqslant i}\mathrm{e}^{-2\bar{\psi}_\ell(t)\psi_\ell(t)} \tag{4.1}$$

which takes into account the string operator present in the definition of the Jordan–Wigner fermions. The action describing the whole reaction–diffusion process reads

$$\begin{aligned}
S_{\mathrm{DP}} = \sum_i \int \mathrm{d}t \Bigg[&\bar{\psi}_i\left(\partial_t + \gamma - \lambda - \left(D + \frac{\lambda}{2}\right)\Delta\right)\psi_i + (\lambda + k - 2D)\bar{\psi}_i\psi_i\bar{\psi}_{i+1}\psi_{i+1} \\
&- (\lambda + k)\bar{\psi}_i\psi_i\xi_{i-1}\partial\psi_i + \lambda\bar{\psi}_i\psi_i\xi_{i-1}\partial\bar{\psi}_i \Bigg] \\
&- \sum_{i,j} \int \mathrm{d}t\,\mathrm{d}t'\Theta(t - t')[-(\lambda + k)\bar{\psi}_i\psi_i\xi_{i-1}\partial\psi_i + \lambda\bar{\psi}_i\psi_i\xi_{i-1}\partial\bar{\psi}_i](t) \\
&\times [-(\lambda + k)\bar{\psi}_j\psi_j\xi_{j-1}\partial\psi_j + \lambda\bar{\psi}_j\psi_j\xi_{j-1}\partial\bar{\psi}_j](t')
\end{aligned} \tag{4.2}$$

with $\partial f_i \equiv \frac{1}{2}(f_{i+1} - f_{i-1})$ and $\Delta f_i \equiv f_{i+1} - 2f_i + f_{i-1}$. Finally, note that we have omitted terms (located at the time slice $t = 0$) describing the initial state of the system.

4.2. Scaling analysis

After rescaling fields and coupling constants we may take

$$S[\bar{\psi}, \psi] = \int dx\, dt[\bar{\psi}(\partial_t + D\sigma - D\partial_x^2)\psi + Dg\psi\bar{\psi}\xi(\partial_x\psi - \partial_x\bar{\psi})]$$

$$+ \int dx\, dy\, dt\, dt'[u_1\{\psi\bar{\psi}\xi\partial_x\psi\}_{(x,t)}\Theta(t-t')\{\psi\bar{\psi}\xi\partial_y\bar{\psi}\}_{(y,t')}$$

$$+u_2\{\psi\bar{\psi}\xi\partial_x\bar{\psi}\}_{(x,t)}\Theta(t-t')\{\psi\bar{\psi}\xi\partial_y\psi\}_{(y,t')}$$

$$+u_3\{\psi\bar{\psi}\xi\partial_x\psi\}_{(x,t)}\Theta(t-t')\{\psi\bar{\psi}\xi\partial_y\psi\}_{(y,t')}$$

$$+u_4\{\psi\bar{\psi}\xi\partial_x\bar{\psi}\}_{(x,t)}\Theta(t-t')\{\psi\bar{\psi}\xi\partial_y\bar{\psi}\}_{(y,t')}] \tag{4.3}$$

as the starting point of the subsequent analysis. The coupling constants are $u_1 = u_2 = -u_3 = -u_4 = (Dg)^2$. We find the field and coupling constants dimensions by power counting in momentum unit:

$$[\psi] = [\bar{\psi}] = \tfrac{1}{2} \qquad [x^2] = [t] = -2 \qquad [g^2] = 1. \tag{4.4}$$

There is an interesting difference with the one-dimensional Reggeon field theory: there the equivalent coupling constant g^2 has dimension $[g^2] = \varepsilon = 4 - d = 3$. It is natural to expect, then, that perturbation will lead to more sensible results than the mere $d \to 1$ extrapolation of the ε-expansion.

In the action equation (4.3) the nonlocal exponential factors

$$\xi(x, t) = \exp\left(-2\int_{-\infty}^x dy\, \bar{\psi}\psi(y, t)\right) \tag{4.5}$$

play a crucial role. Indeed, by construction their presence breaks the space $x \to -x$ symmetry of the action, along with translation invariance. This signals that the right quantities to study in a renormalization perpective are not the usual vertex functions.

Finally, note the absence of the quartic interaction term of the form

$$\int dt \sum_i \bar{\psi}_i\psi_i\bar{\psi}_{i+1}\psi_{i+1}. \tag{4.6}$$

It may readily be seen that this term can be written in the form

$$\int dt \sum_i \bar{\psi}_i\psi_i\bar{\psi}_{i+1}\psi_{i+1} = \int dt \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}\frac{dq_4}{2\pi}$$

$$\times(2\pi)\delta(q_1 + q_2 + q_3 + q_4)(e^{i(q_3+q_4)} - 1 - i(q_3 + q_4))$$

$$\times\bar{\psi}(q_1, t)\psi(q_2, t)\bar{\psi}(q_3, t)\psi(q_4, t) \tag{4.7}$$

which in the small momentum region yields an irrelevant operator of *negative* dimension equal to -1.

4.3. Correlation functions and renormalization

At this stage we have to choose a renormalization scheme. We have explained why the natural choice of examining the divergences of the vertex functions is not adapted here since we are dealing with a theory that neither has translation invariance nor space reflection symmetry. This is due to the anisotropic construction of the fermion operators. In fact the two-point vertex function can still be used to renormalize the fields, the diffusion constant and the mass. As for the coupling constant, we have chosen a particular class of correlation functions in which space arguments are carefully ordered. As could be expected the perturbation series
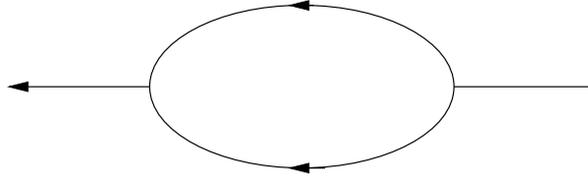
Figure 1. A plain leg denotes a ψ field while an arrowed one denotes a conjugate $\bar{\psi}$. This is a diagram contributing to the renormalization of the propagator. Two of the four internal legs carry a derivative. In Fourier space, the loop integral has to take into account the product of the momenta carried by those legs.

organizes in powers of the dimensionless coupling $g^2/\sqrt{\sigma}$, and our interest lies in the critical regime $\sigma \to 0$. Renormalization is thus needed to extract physical information from the above naive expansion.

We will renormalize the massive theory at zero external frequency and momentum [14]. We define the renormalized fields and parameters by

$$\psi = Z_\psi^{1/2}\psi_R$$
$$\bar{\psi} = Z_\psi^{1/2}\bar{\psi}_R$$
$$D = Z_\psi^{-1}Z_D D_R \tag{4.8}$$
$$(\bar{\psi}\psi) = Z_\sigma^{-1}(\bar{\psi}\psi)_R$$
$$g \equiv Z_D^{-1}Z_\psi^{-1/2}Z_g g_R \sigma_R^{1/4}.$$

The first three Z-factors are determined by the following conditions:

$$\frac{\partial \Gamma_R^{(1,1)}}{\partial(-i\omega)}(0,0) = 1 \qquad \frac{\partial \Gamma_R^{(1,1)}}{\partial k^2}(0,0) = D_R \qquad \Gamma_R^{(1,1)}(0,0) = D_R \sigma_R. \tag{4.9}$$

The one-loop expression for the propagator, calculated after the diagram depicted in figure 1, reads

$$\Gamma^{(1,1)}(k,\omega) = -i\omega\left(1 - \frac{g^2}{2}\int\frac{dq}{2\pi}\frac{q^2}{(q^2+\sigma)^2}\right)$$
$$+ Dk^2\left(1 - g^2\int\frac{dq}{2\pi}\frac{q^2\sigma}{(q^2+\sigma)^3}\right) + D\sigma + g^2\int\frac{dq}{2\pi}\frac{q^2}{q^2+\sigma}. \tag{4.10}$$

An explicit computation of the integrals yields

$$\Gamma^{(1,1)}(k,\omega) = -i\omega\left(1 - \frac{g^2}{8\sqrt{\sigma}}\right) + Dk^2\left(1 - \frac{g^2}{16\sqrt{\sigma}}\right) + D\sigma + g^2\int\frac{dq}{2\pi}\frac{q^2}{q^2+\sigma}. \tag{4.11}$$

In order to renormalize the composite operator $\bar{\psi}\psi$ one notes, as in [15], that a vertex function with a $\bar{\psi}\psi$ insertion is obtained by differentiating the corresponding vertex function with respect to $D\sigma$ and we choose the condition $\Gamma_R^{(1,1;1)}(0,0) = 1$. As for the renormalization of the coupling constant g we may choose to look at the correlation function

$$G^{(1,2)}(x_1, t_1; x_2, t_2, x_3, t_3) \equiv \langle\bar{\psi}(x_1, t_1)\psi(x_2, t_2)\psi(x_3, t_3)\rangle \tag{4.12}$$

with the restriction that $x_2 < x_1 < x_3$. Following the physical explanation proposed by Cardy [15], one understands it is more convenient to look at the the space integral of $G^{(1,2)}$ with the space arguments kept ordered as specified. We now define a quantity very similar to a vertex function: we first continue the latter function correlation function in which the arguments are ordered to unconstrained space arguments. This does allow for an easy calculation of its Fourier transform. We define the associated 1PI function as $\Gamma^{(1,2)}(k_1, k_2, k_3)$ (by convention,

leg 3 carries the derivative ik_3). We emphasize that *this is not* the usual vertex function as defined by the Legendre transform of the logarithm of the partition function. We define the renormalized coupling g_R by

$$\frac{\partial \Gamma_R^{(1,2)}}{\partial(ik_3)}(\{0,0\}) = 2D_R g_R. \tag{4.13}$$

We refer the reader to figure 2 for the corresponding one-loop diagram. A rather tedious calculation yields

$$\frac{\partial \Gamma^{(1,2)}}{\partial(ik_3)}(\{0,0\}) = 2Dg\left(1 - \frac{g^2}{2}\int\frac{dq}{2\pi}\frac{q^4}{(q^2+\sigma)^3} + \frac{5g^2}{4}\int\frac{dq}{2\pi}\frac{q^2}{(q^2+\sigma)^2}\right)$$

$$= 2Dg\left[1 - \frac{7}{32}\frac{g^2}{\sqrt{\sigma}}\right] \tag{4.14}$$

so that the Z-factors read

$$Z_\psi = 1 + \frac{g_R^2}{8} \qquad Z_D = 1 + \frac{g_R^2}{16} \qquad Z_\sigma = 1 + \frac{g_R^2}{4} \qquad Z_g = 1 + \frac{7g_R^2}{32}. \tag{4.15}$$

Denoting the renormalized correlation length by $\xi \equiv \sigma_R^{-1/2}$, the one-loop Wilson functions $\gamma_i \equiv -\xi\frac{d\ln Z_i}{d\xi}, i = \psi, D, \sigma, g$ follow:

$$\gamma_\psi \simeq -\frac{g_R^2}{8} \qquad \gamma_D \simeq -\frac{g_R^2}{16} \qquad \gamma_\sigma \simeq -\frac{g_R^2}{4} \qquad \gamma_g \simeq -\frac{7g_R^2}{32}. \tag{4.16}$$

The β-function reads

$$\beta_g \equiv -\xi\frac{dg_R}{d\xi} = g_R(-\tfrac{1}{2} + \gamma_D + \tfrac{1}{2}\gamma_\psi - \gamma_g) \tag{4.17}$$

hence, to one-loop,

$$\beta_g = -g_R(\tfrac{1}{2} - \tfrac{3}{32}g_R^2). \tag{4.18}$$

The renormalization group flow then has a single stable fixed point $g_R^{*2} = \frac{16}{3}$. The critical exponents can be expressed in terms of the Wilson functions at the fixed point:

$$\begin{aligned} \eta &= \gamma_\psi^* \simeq -\tfrac{2}{3} \\ z &= 2 - \gamma_D^* + \gamma_\psi^* \simeq \tfrac{5}{3} \\ \nu^{-1} &= 2 - \gamma_D^* + \gamma_\sigma^* \simeq \tfrac{4}{3}. \end{aligned} \tag{4.19}$$

The above numerical values are in reasonable agreement with the precise numerical estimates of Dickman [19] ($\eta = -0.405$, $z = 1.767$, $\nu = 1.142$).

5. Conclusions

5.1. Comparison with existing approaches

Of course the method we have presented suffers two obvious drawbacks. First, the numerical values of the critical exponents poorly compare with those given by numerical simulations. This is due to the absence of a small parameter validating the first-order loop expansion. From that point of view the situation is quite similar to the low-dimensional extrapolation of ε-expansions. A related question concerns the convergence of the series expansion of the exponents in terms of the fixed-point renormalized coupling. Second, we lack a decisive argument in favour of the renormalizability of the theory, a property we trust in, but that we were unable to establish on a rigorous basis. Arguments in favour of the existence of
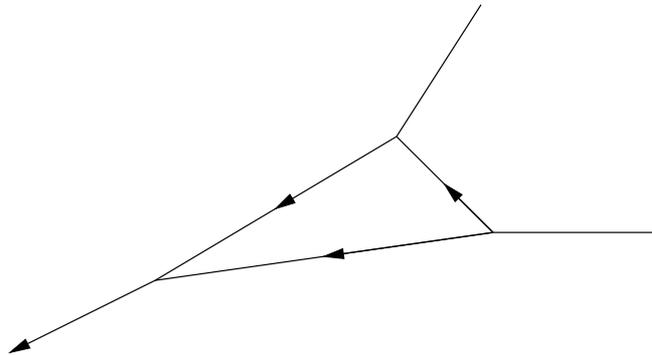
Figure 2. One of the diagrams entering the renormalization of g. It involves u_1 and g.

consistent renormalization rely on the locality of the original spin theory. The scheme that we have introduced, based upon the study of space-ordered correlation functions (and their related loop integrals), must now be extended to all orders. However we emphasize that we have built an entirely new and systematic analytic technique that provides to our knowledge the only alternative to ε-expansion of the Reggeon field theory and to short time series expansions. The approximation scheme we have developed has, in principle, a wide range of applicability and exhibits many analogies with its bosonic counterpart. Its main advantage is that it naturally incorporates the simplification induced by the one-dimensional topology. As a final comment, we find it amazing that the present field theory has appeared in a slightly different form in a quite different context [15].

5.2. Prospects

We have also applied the above formalism [16] to the much studied annihilation reaction $A + A \rightarrow \emptyset$. The methods of integrable spin chains [9, 10] work at a particular diffusion constant-annihilation rate ratio (which corresponds to a free fermion Hamiltonian equation (2.6)) and leads, as expected, to a $1/\sqrt{t}$ decay of the particle density. The present method very easily allows us to extend this result to any value of the parameters.

There exists another class of reaction–diffusion problems, namely branching and annihilating random walks [17], in which the formalism we have developed should profitably apply. We expect it to provide a better qualitative picture in one dimension than previous attempts [17] using bosonic theory down fromthe upper critical dimension. It also remains to be seen how the formalism can be extended to cope with several species reaction–diffusion problems (e.g. [18]) for which no analytical nor numerical results exist in one dimension. Our tool should prove useful in the study of those problems, where other techniques have, up to now, failed.

Acknowledgments

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