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Global persistence in directed percolation

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Abstract. We consider a directed percolation process at its critical point. The probability that the deviation of the global order parameter with respect to its average has not changed its sign between 0 and t decays with t as a power law. In space dimensions $d \geqslant 4$ the *global persistence exponent* θ_p that characterizes this decay is $\theta_p = 2$ while for $d < 4$ its value is increased to first order in $\varepsilon = 4-d$. Combining a method developed by Majumdar and Sire with renormalization group techniques we compute the correction to θ_p to first order in ε. The global persistence exponent is found to be a new and independent exponent. Finally we compare our results with existing simulations.

1. Motivations

1.1. Directed percolation

At the initial time A particles are placed randomly with density ρ_0 on the sites of a d-dimensional hypercubic lattice. They perform independent simple random walks with a diffusion contant λ. Multiple occupancy is allowed. The A particles undergo three reaction processes: coagulation upon encounter at a rate k, branching at a rate k', spontaneous death at a rate γ,

$$\begin{aligned} A + A &\xrightarrow{k} A \\ A &\xrightarrow{k'} A + A \\ A &\xrightarrow{\gamma} \emptyset. \end{aligned} \tag{1.1}$$

As the branching rate k' is decreased below a threshold value k'_c (equal to γ in mean field), the steady state of this system exhibits a continuous transition from a state in which a finite positive density of A's survive indefinitely to an absorbing A-free state. The order parameter of the transition is ρ_A, the average of the local density of A's.

We have used the language of the Schlögl reaction–diffusion process to describe directed percolation as in [2]. Various alternative formulations exist [3, 4]. In directed percolation itself one usually starts from a single germ located at site $\boldsymbol{x} = \mathbf{0}$ at the initial time $t = 0$, which evolves according to the rules equation (1.1). The interest is then focused on the structure of the cluster of visited sites in the $(d+1)$-dimensional $(\boldsymbol{x}, t)$-space. Nevertheless the scope of directed percolation reaches far beyond chemical kinetics, as an overwhelmingly

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large class of nonequilibrium systems possessing a phase transition in their steady state fall in the same universality class (cellular automata in information theory, surface growth in which the interface has the same scaling properties as the hull of a directed percolation cluster, several autocatalytic reactions). This makes the process equation (1.1) a paradigm for nonequilibrium systems with a transition in their steady state. Our knowledge of the behaviour of the system in the steady state and during the relaxation stages rests on numerical simulations (in low space dimensions, $d = 1, 2$) and on analytical techniques (short time series expansions in $d = 1, 2$, renormalization group in $d = 4 - \varepsilon$). The critical regime is characterized by a set of three independent exponents: the dynamical exponent z, the anomalous dimension of the order parameter η and the correlation length exponent ν. Scaling laws for ρ_A can be extracted from special cases of

$$\rho_A(t, k' - k'_c, \rho_0) = b^{-\frac{d+\eta}{2}} \mathcal{F}(b^{1/\nu}|k' - k'_c|, b^{-z}t, b^{\frac{d-\eta}{2}}\rho_0) \tag{1.2}$$

which holds in the limit $b \to \infty$ with the arguments of $\mathcal{F}$ fixed. Similar scaling relations exist for correlation functions.

1.2. Global persistence

In this paper we wish to focus on a property that cannot be deduced from the knowledge of the scaling properties of correlation functions alone. We first define the deviation of the global time-dependent order parameter with respect to its average:

$$\Psi(t) \equiv \lim_{L\to\infty} L^{-d/2} \sum_{\boldsymbol{x}\in L^d} [n_A(\boldsymbol{x}, t) - \langle n_A(\boldsymbol{x}, t)\rangle]. \tag{1.3}$$

In equation (1.3) we denote by $n_A(\boldsymbol{x}, t)$ the number of A particles at site $\boldsymbol{x}$ at time t, in a particular realization of the reaction–diffusion process. The brackets $\langle \ldots \rangle$ denote an average with respect to the set of microscopic realizations consistent with the initial conditions, the rules equation (1.1) and the diffusion. The variable Ψ has a simple interpretation. When one looks at a particular microscopic realization of the system Ψ states how far the total number of particles deviates from a typical realization.

We define the *global persistence* probability as the probability that Ψ remain of constant sign between 0 and t. Similar quantities have been considered (see [1, 5–7]) in critical dynamics of magnetic systems, Ψ simply being the total magnetization. There it was shown that, following a quench from a high-temperature disordered state to the critical point, the global persistence probability decays with time as a power law characterized by a universal exponent θ_p. In critical dynamics the persistence probability is a quantity that appears naturally in the description of the system while it relaxes to its equilibrium state. Our motivation for this work lies in the lack of both qualitative and analytical picture of the onset of long-range correlations in nonequilibrium systems relaxing to their steady state. We believe that the knowledge of the global persistence probability will shape our picture of the way the system organizes at criticality.

This paper is organized as follows. In section 2 we recall the well known correspondence between directed percolation and field theory. Following Majumdar and Sire [1], it is possible to obtain the global persistence probability from a careful analysis of the autocorrelation of the global order parameter. This analysis is performed in great detail in sections 3 and 4. In section 5 we turn to the explicit calculation of the persistence exponent. In our conclusion we compare our results with existing simulations.

2. Field theoretic formulation

There are several ways of mapping directed percolation onto a field theory [2, 3]. The resulting field theory involves a field ψ whose average is the local density of A individuals, and a conjugate field $\bar{\psi}$; dropping terms irrelevant in the vicinity of the upper critical dimension $d_c = 4$ the corresponding action reads

$$S[\psi, \bar{\psi}] = \int \mathrm{d}^d x \, \mathrm{d}t \left[\bar{\psi}(\partial_t + \lambda(\sigma - \Delta))\psi + \frac{\lambda g}{2}\psi\bar{\psi}(\psi - \bar{\psi}) - \rho_0 \delta(t)\bar{\psi} \right]. \tag{2.1}$$

The parameter g can be expressed in terms of the coarse grained analogues of the original reaction rates k, k', γ and the mass in the propagator is the coarse grained analogue of $\gamma - k'$. The action equation (2.1) is the starting point of the subsequent analysis. Renormalization group techniques allow us to focus on scaling laws close to or at criticality, during the relaxation process or in the steady state. Hereafter, as we shall eventually focus on phenomena taking place at criticality, we set $\sigma = 0$. We now summarize a few well known results on the renormalization of the action equation (2.1) that can be found, e.g. in [8].

One first defines renormalized parameters and fields as follows

$$\psi = \sqrt{Z}\psi_{\mathrm{R}} \qquad \bar{\psi} = \sqrt{Z}\bar{\psi}_{\mathrm{R}} \qquad \lambda = Z^{-1}Z_\lambda \lambda_{\mathrm{R}} \qquad \frac{g^2}{(8\pi)^{d/2}} = Z_\lambda^{-2} Z^{-1} Z_u u \mu^\varepsilon \tag{2.2}$$

where μ is a momentum scale. From the one-loop expression of the two- and three-point vertex functions one deduces the values of the Z-factors using dimensional regularization and the minimal subtraction scheme. They read

$$Z = 1 + \frac{u}{\varepsilon} \qquad Z_\lambda = 1 + \frac{u}{2\varepsilon} \qquad Z_u = 1 + \frac{8u}{\varepsilon}. \tag{2.3}$$

The β-function has the one-loop expression

$$\beta_u \equiv \mu \frac{\mathrm{d}u}{\mathrm{d}\mu} = u(-\varepsilon + 2\gamma_\lambda + \gamma - \gamma_u) = u(-\varepsilon + 6u) \tag{2.4}$$

where we have introduced the Wilson functions $\gamma_i \equiv \mu \frac{\mathrm{d}\ln Z_i}{\mathrm{d}\mu}$, $i = \emptyset, \lambda, u$. The β-function has a stable nontrivial fixed point $u^\star = \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2)$. Critical exponents are then obtained from linear combinations of the $\gamma_i(u^\star)$, e.g. $z = 2 - \gamma_\lambda^\star + \gamma^\star$ and $\eta = \gamma^\star$.

We find it convenient to shift ψ by its mean-field expression

$$\psi_{\mathrm{mf}}(t) = \frac{\rho_0}{1 + \frac{\lambda g}{2}\rho_0 t}. \tag{2.5}$$

Therefore the action expressed in terms of the fields $\phi \equiv \psi - \psi_{\mathrm{mf}}$ and $\bar{\phi} \equiv \bar{\psi}$ reads

$$S[\phi, \bar{\phi}] = \int \left[\bar{\phi} \left(\partial_t + \lambda \left(\frac{g\rho_0}{1 + \frac{\lambda g}{2}\rho_0 t} - \Delta \right) \right) \phi - \frac{\lambda g \rho_0}{2(1 + \frac{\lambda g}{2}\rho_0 t)} \bar{\phi}^2 + \frac{\lambda g}{2}\phi\bar{\phi}(\phi - \bar{\phi}) \right]. \tag{2.6}$$

We have thus eliminated the initial term localized at $t = 0$. We will use the following notation: $G^{(n,m)}$ denotes the $(n + m)$-point correlation function involving n fields ϕ and m fields $\bar{\phi}$, as defined in equation (4.7), and $W^{(n,m)}$ denotes its connected counterpart. The basic ingredients for a perturbative expansion are the free propagator G and the free

correlator C, defined by the zero-loop expression of $G^{(1,1)}$ and $G^{(2,0)}$, respectively. We shall need the large time behaviour of G and C:

$$G(\boldsymbol{k};t',t) = \Theta(t'-t)\left(\frac{t}{t'}\right)^2 \exp[-\lambda(\boldsymbol{k}^2+\sigma)(t'-t)] \tag{2.7}$$

$$C(\boldsymbol{k};t,t') = \frac{2}{t_>^2 t_<^2}\mathrm{e}^{-\lambda\boldsymbol{k}^2(t_<+t_>)}\frac{1}{(2\lambda\boldsymbol{k}^2)^4}[6(1-\mathrm{e}^{2\lambda\boldsymbol{k}^2 t_<}) + 6(2\lambda\boldsymbol{k}^2 t_<)\mathrm{e}^{2\lambda\boldsymbol{k}^2 t_<} - 3(2\lambda\boldsymbol{k}^2 t_<)^2\mathrm{e}^{2\lambda\boldsymbol{k}^2 t_<} + (2\lambda\boldsymbol{k}^2 t_<)^3\mathrm{e}^{2\lambda\boldsymbol{k}^2 t_<}] \tag{2.8}$$

in which we have set $t_< \equiv \min\{t,t'\}$ and $t_> \equiv \max\{t,t'\}$. Note that the dependence on the initial density ρ_0 has disappeared, this is because we are focusing on times large with respect to the timescale set by ρ_0. We now have the building blocks for a perturbation expansion of the expectation values of time-dependent observables.

3. Autocorrelation function

Our aim is to find the one-loop correction to the function $\mathcal{C}(t,t')$ defined by

$$\mathcal{C}(t,t') \equiv \int \mathrm{d}^d r\, W^{(2,0)}(\boldsymbol{r},t;\boldsymbol{0},t') = W^{(2,0)}(\boldsymbol{k}=\boldsymbol{0};t,t') \tag{3.1}$$

which is merely the autocorrelation function of the field ψ. In order to determine $\mathcal{C}(t,t')$ we carry out a perturbation expansion in powers of the coupling constant g. The first term of this expansion is of course $C(\boldsymbol{k}=\boldsymbol{0};t,t')$. The first nontrivial corrections come in six pieces, each depicted by a one-loop connected Feynman diagram shown in figure 1. The explicit calculation of these diagrams combined with equations (2.2) and (2.3) allows one to determine the renormalized autocorrelation function

$$\mathcal{C}_\mathrm{R}(t,t') = Z^{-1}\mathcal{C}(t,t'). \tag{3.2}$$

Using again the shorthand notations $t_< = \min\{t,t'\}$, $t_> = \max\{t,t'\}$, we find

$$\mathcal{C}_\mathrm{R}(t,t') = \frac{1}{2}(\lambda t_< \mu^2)^{\frac{\varepsilon}{12}}\left(\frac{t_<}{t_>}\right)^{2-\frac{\varepsilon}{4}} A\left[1-\frac{\varepsilon}{6}F(t_>/t_<)+\mathcal{O}(\varepsilon^2)\right] \tag{3.3}$$

which holds for $t_<, t_>$ large with $t_>/t_<$ finite. In equation (3.3) the amplitude A reads

$$A = 1+\frac{\varepsilon}{6}\left(\frac{1457}{600}+\frac{\pi^2}{20}-\frac{96}{25}\ln 2\right)+\mathcal{O}(\varepsilon^2) \tag{3.4}$$

and the function F has the expression

$$\begin{aligned} F(x) = {} & -\frac{19}{30x}+\frac{11}{6}+\frac{\pi^2}{10}-\frac{96}{25}\ln 2+\frac{12}{25}x-\frac{7}{50}x^2-\frac{77}{50}x^3 \\ & +\ln(1-x^{-1})\left[\frac{101}{25}-5x+x^2+x^3-\frac{26}{25}x^4-\frac{1}{25}x^5+\frac{1}{25x}\right] \\ & +\ln(1+x^{-1})\left[\frac{23}{50}+x+x^2+x^3+\frac{23}{50}x^4-\frac{1}{25}x^5-\frac{1}{25x}\right] \\ & +2\ln^2 x+3\ln x\ln(1-x^{-1})+4\mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-x^{-1}) \\ & -\tfrac{3}{20}(x^4-1)\mathrm{Li}_2(x^{-2})-\tfrac{3}{5}\mathrm{Li}_2(x^{-1}) \end{aligned} \tag{3.5}$$

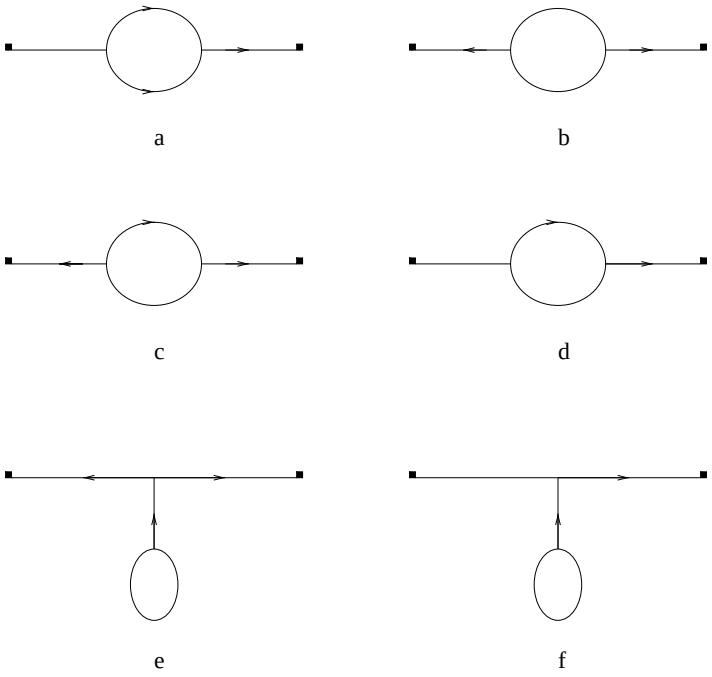

Figure 1. One-loop diagrams involved in the expression of $\mathcal{C}(t,t')$. Note that diagrams (a), (d), (f) are the only one-loop contributions to $\langle\bar{\psi}(\mathbf{0},0)\psi(\mathbf{0},t)\rangle$. We have used the following graphical conventions: a plain leg denotes a ψ field, an arrowed leg a $\bar{\psi}$ field; the times of the two external ψ legs are fixed, which we have indicated by a black square.

where $\mathrm{Li}_2(x) = -\int_0^x \mathrm{d}t\ \ln(1-t)/t$ is the dilogarithm function. The limiting behaviour of F is found to be

$$\begin{aligned} &F(\infty) = \tfrac{8329}{1200} - \tfrac{2}{5}\pi^2 - \tfrac{96}{25}\ln 2 = 0.3313\ldots \\ &F(x) \stackrel{x\to 1}{=} -2(x-1)\ln(x-1) + \mathcal{O}(x-1). \end{aligned} \tag{3.6}$$

In the appendix we have listed the individual contributions to $\mathcal{C}(t,t')$ arising from the corresponding Feynman diagrams.

4. Short time expansion

The result of the previous section equation (3.3) for $\mathcal{C}_{\mathrm{R}}(t',t)$ holds for all times t and t', with t/t' finite, but the limit $t \ll t'$ is singular. In this section we show that for $t \ll t'$ the autocorrelation function $\mathcal{C}_{\mathrm{R}}(t',t)$ displays power law behaviour with respect to both time arguments and determine the corresponding exponents. In this limit the random variable $\Psi(t)$ becomes a Markovian process for which the persistence exponent may be expressed in terms of well known critical exponents. In the case of the Ising model the Markovian approximation for θ_p is already close to the values obtained by simulations [6, 5].

An appropriate method to study the correlation function for $t \ll t'$ is the short time expansion (STE) of the field $\psi(\boldsymbol{r},t)$ in terms of operators located at the 'time surface' $t = 0$. Since the Gaussian propagator and correlator are of the order t^2 for $t \to 0$ we expect

that the leading term in the STE is the second time derivative $\ddot{\psi}$ of ψ, i.e.

$$\psi(\boldsymbol{r},t)-\langle\psi(\boldsymbol{r},t)\rangle = c(t)\ddot{\psi}(\boldsymbol{r},0)+\cdots. \tag{4.1}$$

(For $t=0$ the second time derivative of the response field $\bar{\psi}$ is equivalent to $\ddot{\psi}$.) The function $c(t)$ is a power of t which can be obtained from the difference of the scaling dimensions of ψ and $\ddot{\psi}$. Naively, $c(t)\sim t^2$.

To compute the scaling dimension of $\ddot{\psi}$ in an ε-expansion the first idea is to determine the additional renormalization that is necessary to render correlation functions with $\ddot{\psi}$ insertions finite. For instance the one-loop expression of $\langle\ddot{\psi}(\boldsymbol{q}=\boldsymbol{0},0)\psi(\boldsymbol{q}=\boldsymbol{0},t)\rangle$ reads

$$\langle\ddot{\psi}(\boldsymbol{0},0)\psi(\boldsymbol{0},t)\rangle = \frac{1}{t^2}\left[1+\frac{g^2}{(8\pi)^{d/2}}(\lambda t)^{\varepsilon/2}\left(\frac{3}{\varepsilon}+\mathcal{O}(1)\right)+\mathcal{O}(g^4)\right]. \tag{4.2}$$

It results from the contribution of diagrams (a), (d), (f) in figure 1. The pole in ε may be absorbed into an additional renormalization of $\ddot{\psi}(0)$:

$$\ddot{\psi}(0) = (ZZ_0)^{1/2}\ddot{\psi}_{\mathrm{R}}(\boldsymbol{r},0) \qquad Z_0 = 1+\frac{4u}{\varepsilon}. \tag{4.3}$$

At this stage we can give the scaling dimension of $\ddot{\psi}(0)$:

$$d(\ddot{\psi}) = \frac{d+\eta}{2}+2z+\frac{1}{2}\gamma_0(u^\star) \tag{4.4}$$

where $\gamma_0=-4u+\mathcal{O}(u^2)$ is the Wilson function associated to Z_0, and obtain the additional anomalous dimension of $\ddot{\psi}(0)$ to first order in ε.

Fortunately, it is possible to express this dimension to every order in ε in terms of other critical exponents. For the initial density $\rho_0=\infty$ there is a similarity between directed percolation and the semi-infinite Ising model at the normal transition (i.e. for infinite surface magnetization). In the latter case the short-distance expansion of the order parameter field near the surface is governed by the stress tensor [9, 10]. Due to the translational invariance of the bulk Hamiltonian the stress tensor requires no renormalization. Here we look for an initial field which remains unrenormalized as a consequence of the translational invariance (with respect to time) of the stationary state.

Our argument applies to any dynamic field theory defined by a dynamic functional of the form

$$S[\psi,\bar{\psi}] = \int_0^\infty \mathrm{d}t\int \mathrm{d}^d r\,(\bar{\psi}\partial_t\psi - T[\psi,\bar{\psi}]). \tag{4.5}$$

We assume that ψ satisfies the sharp initial condition $\psi(\boldsymbol{r},0)=\rho_0$. For directed percolation we have

$$T[\psi,\bar{\psi}] = -\lambda\bar{\psi}(\sigma-\Delta)\psi-\frac{\lambda g}{2}\psi\bar{\psi}(\psi-\bar{\psi}). \tag{4.6}$$

Correlation functions may be written in the form

$$G^{(n,m)}(\{\boldsymbol{r},t\}) = \int \mathcal{D}[\psi,\bar{\psi}]\prod_{i=1}^{m}\bar{\psi}(\bar{\boldsymbol{r}}_i,\bar{t}_i)\prod_{j=1}^{n}\psi(\boldsymbol{r}_j,t_j)\exp(-S[\psi,\bar{\psi}]) \tag{4.7}$$

where the functional integral runs over all histories $\{\psi,\bar{\psi}\}$ which satisfy the initial condition.

We now introduce a new time variable $t\to t'=t+a(t)$ (with $\dot{a}(t)>-1$ to maintain the time order) and the transformed fields $\bar{\psi}'$ and ψ' with $\bar{\psi}'(t)=\bar{\psi}(t')$ and $\psi'(t)=\psi(t')$. At lowest order in $a(t)$ the dynamic functional becomes

$$S[\psi,\bar{\psi}] = S[\psi',\bar{\psi}']-\int \mathrm{d}t\int \mathrm{d}^d r\,\dot{a}(t)T[\psi',\bar{\psi}'] \tag{4.8}$$

where $a(0) = 0$ has been assumed.

Performing the time shift in the correlation function $G^{(n,m)}$ and comparing the terms of first order in $a(t)$ on both sides of equation (4.7) one finds

$$\left(\sum_{i=1}^{m} a(\bar{t}_i)\frac{\partial}{\partial \bar{t}_i} + \sum_{j=1}^{n} a(t_j)\frac{\partial}{\partial t_j}\right) G^{(n,m)}(\{\boldsymbol{r}, t\}) = \left\langle \bar{\psi}(\bar{\boldsymbol{r}}_1, \bar{t}_1)\cdot \ldots \psi(\boldsymbol{r}_n, t_n) \int \mathrm{d}t \int \mathrm{d}^d r\, \dot{a}(t) T[\psi, \bar{\psi}] \right\rangle. \tag{4.9}$$

Here the angular brackets indicate the average with respect to the weight $\exp(-S[\psi, \bar{\psi}])$. We may choose

$$a(t) = a_0(1 - \mathrm{e}^{-vt}) \tag{4.10}$$

to obtain in the limit $v \to \infty$

$$\left(\sum_{i=1}^{m} \frac{\partial}{\partial \bar{t}_i} + \sum_{j=1}^{n} \frac{\partial}{\partial t_j}\right) G^{(n,m)}(\{\boldsymbol{r}, t\}) = \left\langle \bar{\psi}(\bar{\boldsymbol{r}}_1, \bar{t}_1)\cdot \ldots \psi(\boldsymbol{r}_n, t_n) \int \mathrm{d}^d r\, T_+ \right\rangle \tag{4.11}$$

where T_+ denotes the operator $T[\psi, \bar{\psi}]$ in the limit $t \to 0^+$. (We have assumed that all time arguments of the correlation function are nonzero.)

This result shows that T_+ remains unrenormalized to every order of the perturbation theory. Therefore its scaling dimension is given by $d(T_+) = d + z$. At the upper critical dimension $d_c = 4$ we find $d(T_+) = d(\ddot{\psi}) = 6$. In fact, one can show that T_+ and $\ddot{\psi}(0)$ differ for $\rho_0^{-1} = 0$ only by a constant prefactor. To see this we express $T[\psi, \bar{\psi}]$ in terms of the shifted field $\phi = \psi - \psi_{\mathrm{mf}}$. Since $\phi(t), \bar{\psi}(t) \sim t^2$ for $t \to 0$ while $\psi_{\mathrm{mf}}(t) \sim t^{-1}$ only the term $-(\lambda g/2)\psi_{\mathrm{mf}}^2 \bar{\psi}$ contributes. Thus $T_+ \sim \ddot{\bar{\psi}}(0) \sim \ddot{\psi}(0)$, and the STE in equation (4.1) becomes

$$\psi(\boldsymbol{r}, t) - \langle \psi(\boldsymbol{r}, t)\rangle = c(t) T_+(\boldsymbol{r}) + \cdots \tag{4.12}$$

with

$$c(t) \sim t^{-(d(\psi) - d(T_+))/z} = t^{-((d+\eta)/2 - (d+z))/z}. \tag{4.13}$$

Combining the STE with the general scaling form of the autocorrelation function one obtains

$$\mathcal{C}_{\mathrm{R}}(t', t) \sim t'^{-\eta/z} \left(\frac{t}{t'}\right)^{1+(d-\eta)/(2z)} \tag{4.14}$$

which holds for $t/t' \to 0$.

5. Global persistence

5.1. A detour via the Ornstein–Uhlenbeck process

Let $X(\tau)$ be a Gaussian process with the following autocorrelation function

$$\langle X(\tau) X(\tau')\rangle = \mathrm{e}^{-\theta_p^{(0)}|\tau' - \tau|} \tag{5.1}$$

for τ, τ' large (but arbitrary $\tau - \tau'$). The random variable X is thus a Gaussian stationary Markov process (of unit variance). It satisfies a Langevin equation

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} = -\theta_p^{(0)} X(\tau) + \zeta(\tau) \qquad \theta_p^{(0)} > 0 \tag{5.2}$$

where ζ is Gaussian white noise:

$$\langle \zeta(\tau)\zeta(\tau')\rangle = 2\theta_p^{(0)}\delta(\tau-\tau'). \tag{5.3}$$

Therefore X is an Ornstein–Uhlenbeck process. For such a process the probability that X be positive between 0 and τ decays exponentially as

$$\mathrm{Prob}\{\forall\tau' \in [0,\tau],\, X(\tau') > 0\} \propto \mathrm{e}^{-\theta_p^{(0)}\tau}. \tag{5.4}$$

These are standard results.

5.2. Expansion around an Ornstein–Uhlenbeck process

We now consider a Gaussian stationary process $X(\tau)$ which has the autocorrelation function

$$\langle X(\tau)X(\tau')\rangle = \mathrm{e}^{-\theta_p^{(0)}|\tau'-\tau|} + \epsilon f(\tau'-\tau) \tag{5.5}$$

with $f(0) = 0$ and $\epsilon \ll 1$. Then X is not a Markovian process. Majumdar and Sire [1] have shown how to evaluate the probability that X is positive between 0 and τ to first order in ϵ. They found

$$\mathrm{Prob}\{\forall\tau' \in [0,\tau],\, X(\tau') > 0\} \propto \mathrm{e}^{-\theta_p\tau} \tag{5.6}$$

where

$$\theta_p = \theta_p^{(0)}\left[1 - \epsilon\frac{2\theta_p^{(0)}}{\pi}\int_0^\infty \mathrm{d}\tau\, \frac{f(\tau)}{(1-\exp(-2\theta_p^{(0)}\tau))^{3/2}} + \mathcal{O}(\epsilon^2)\right]. \tag{5.7}$$

Hakim [11] extended this result to $\mathcal{O}(\epsilon^2)$.

5.3. Application to the global order parameter

At any fixed time t there exists a dynamical correlation length $\xi \sim t^{1/z}$ such that the system may be considered as a collection of effectively independent blocks of linear size ξ. Hence Ψ is the sum of $(L/\xi)^d$ independent degrees of freedom, which is a Gaussian variable in the limit $L \to \infty$. We introduce the random variable

$$X(\tau) \equiv \Psi(\mathrm{e}^\tau)/\sqrt{\mathrm{var}\,\Psi(\mathrm{e}^\tau)} \tag{5.8}$$

which has the autocorrelation function

$$\langle X(\tau)X(\tau')\rangle = \mathrm{e}^{-\theta_p^{(0)}|\tau'-\tau|} - \frac{\varepsilon}{6}\mathrm{e}^{-\theta_p^{(0)}|\tau'-\tau|}F(\mathrm{e}^{|\tau'-\tau|}) \tag{5.9}$$

with, after equations (4.14) and (5.1)

$$\theta_p^{(0)} = 1 + \frac{d}{2z}. \tag{5.10}$$

Thus X is a Gaussian stationary process. We are now in a position to apply to X the result of section 5.2. Substitution of equations (5.9) and (5.10) into equation (5.7) yields

$$\theta_p = \theta_p^{(0)}\left[1 + \frac{2\varepsilon}{3\pi}\mathcal{I} + \mathcal{O}(\varepsilon^2)\right] \tag{5.11}$$

where the integral

$$\mathcal{I} \equiv \int_1^\infty \mathrm{d}x\, \frac{x^3 F(x)}{(x^4-1)^{3/2}} \tag{5.12}$$

has the analytic expression

$$\begin{aligned}\mathcal{I} = {} & \frac{13}{200} - \frac{9C}{20} - \frac{91\pi}{200} - \frac{3\pi^2}{16} - \frac{9\pi}{80}\ln 2 + \frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}}\left[\frac{\pi}{4} - \frac{1}{8}\ln 2 + \frac{41}{80}\right] \\ & + \frac{\sqrt{2\pi^3}}{\Gamma(\frac{1}{4})^2}\left[-\frac{77\pi}{200} + \frac{23}{12} - \frac{3}{4}\ln 2\right] + \frac{1}{50}\,{}_3\mathrm{F}_2\left(\frac{1}{4}, 1, 1; \frac{5}{4}, \frac{3}{2}; 1\right) \\ & - \frac{6}{5}\frac{\sqrt{2\pi^3}}{\Gamma(\frac{1}{4})^2}\,{}_3\mathrm{F}_2\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{5}{4}, \frac{5}{4}; 1\right) - \frac{1}{30}\frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}}\,{}_3\mathrm{F}_2\left(\frac{3}{4}, 1, \frac{5}{4}; \frac{7}{4}, \frac{7}{4}; 1\right) \\ & + \frac{\sqrt{2\pi^3}}{\Gamma(\frac{1}{4})^2}\,{}_3\mathrm{F}_2\left(\frac{1}{2}, \frac{3}{4}, 1; \frac{5}{4}, \frac{3}{2}; 1\right) - \frac{1}{240}\frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}}\,{}_3\mathrm{F}_2\left(\frac{1}{4}, 1, 1; \frac{7}{4}, 2; 1\right) \\ & - \frac{1}{150}\frac{\sqrt{2\pi^3}}{\Gamma(\frac{1}{4})^2}\,{}_3\mathrm{F}_2\left(\frac{3}{4}, 1, \frac{3}{2}; \frac{9}{4}, \frac{5}{2}; 1\right) + \frac{1}{2}\,{}_3\mathrm{F}_2\left(\frac{3}{4}, 1, 1; \frac{3}{2}, \frac{7}{4}; 1\right) \\ & - \frac{3}{8}\frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}}\,{}_3\mathrm{F}_2\left(\frac{1}{4}, \frac{1}{2}, 1; \frac{3}{4}, \frac{3}{2}; 1\right) \\ = {} & 0.630237\ldots \end{aligned} \tag{5.13}$$

where C denotes Catalan's constant and ${}_3\mathrm{F}_2$ the hypergeometric function of order $(3, 2)$. The final result reads

$$\theta_p = \theta_p^{(0)}(1 + 0.134\varepsilon + \mathcal{O}(\varepsilon^2)) = 2 + 0.059\varepsilon + \mathcal{O}(\varepsilon^2). \tag{5.14}$$

6. Discussion

6.1. Comparison with existing simulations

Recently Hinrichsen and Koduvely [12] performed a numerical study of one-dimensional directed percolation in order to determine the asymptotic behaviour of the global persistence probability. In terms of the variable Ψ defined in equation (1.3), they found the following results. For the probability that Ψ remain *negative* between 0 and t they indeed found a power law decay characterized by a universal exponent θ_p that has a numerical value $\theta_p \gtrsim 1.50$. However, they found an exponential decay for the probability that Ψ remain *positive* between 0 and t. We have found that the global persistence probability decays algebraically *irrespective* of the sign of Ψ. The interpretation for such an asymmetry in the simulations can be traced back in finite-size effects, which impair the accuracy of the measurement of the persistence exponent. On the one hand, the global persistence exponent is well defined in the regime in which the system has lost the memory of the initial condition. This regime takes place for times t such that $t^{\frac{d-\eta}{2z}}\rho_0 \gg 1$. On the other hand, Ψ is well defined in the limit of infinitely large systems, and then it is the sum of a large number of effectively independent contributions, which, on a lattice of size L, forces $L \gg \xi \sim t^{1/z}$. Hence, in a numerical simulation care must be taken to ensure that the double limit $L^z \gg t \gg \rho_0^{-2z/(d-\eta)}$ is satisfied. The simulations in [12] fulfil the lower bound but not the upper one. This has led the authors to a lower bound on θ_p. Finally, in mapping the random process $\Psi(t)$ to $X(\tau)$ we have assumed that the time interval under consideration contains only times large compared with ρ_0^{-1} so that the regime in which X is stationary be reached. Strictly speaking, we should have defined the persistence probability over a time interval $[t_0, t]$, with $t \gg t_0 \gg \rho_0^{-1}$. In a simulation the choice $t_0 = 0$ leads

to a persistence probability that enters the asymptotic regime after times t very large with respect to ρ_0^{-1}.

6.2. Some speculations

It is interesting to use the ε-expansion to speculate on the numerical value of θ_p in low space dimension. We define the *improved* value of θ_p, which we denote by θ_p^{spec}, by the product of the actual value of $\theta_p^{(0)}$ deduced from numerical simulations, and the $\mathcal{O}(\varepsilon)$ correction given by equation (5.7). We use recent simulations results of one- and two-dimensional directed percolation carried out by Lauritsen *et al* [13]

d	z	ν	η	$\theta_p^{(0)}$	θ_p^{spec}	θ_p
1	1.581	1.097	-0.496	1.316	1.8	$\geqslant 1.50$
2	1.764	0.734	-0.409	1.567	2.0	$-$
$4-\varepsilon$	$2-\frac{\varepsilon}{12}$	$\frac{1}{2}+\frac{\varepsilon}{16}$	$-\frac{\varepsilon}{6}$	$2-\frac{5\varepsilon}{24}$	$-$	$2+0.059\varepsilon$
$\geqslant 4$	2	$\frac{1}{2}$	0	2	2	2

(6.1)

In the first two lines the exponents z, ν and η are taken from [13]; the value of $\theta_p^{(0)}$ was obtained using the hyperscaling relation $\theta_p^{(0)} = 1 + d/2z$, and that of θ_p in $d = 1$ is taken from [12]. Of course the column θ_p^{spec} gives but a qualitative estimate of the true θ_p that is supposedly closer to it than that obtained by the crude ε-expansion. These predictions certainly have to be tested against numerical simulations.

6.3. Final comments

We would like to add some comments on table (6.1). We find that $\theta_p > 2$ in $d < 4$ which states that both the average and the variance of the time during which Ψ is of constant sign are finite. The average time depends on a parameter which has the dimension of time, but there is no timescale left in our treatment of the persistence probability. Therefore the average time must depend on a microscopic scale or ρ_0^{-1}, which we have treated as a microscopic scale. This may also explain why the average time is infinite for critical dynamics: for zero initial magnetization a timescale such as ρ_0^{-1} does not exist. In critical dynamics one could also define a persistence exponent for the critical relaxation from an initial state with nonzero local magnetization. In that case the persistence exponent would read in the Markovian approximation $\theta_p^{(0)} = 1 + d/(2z) > 1$ (as for the problem we have treated in this paper).

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Appendix

We introduce the notations $t_< \equiv \min\{t, t'\}$ and $t_> \equiv \max\{t, t'\}$ and the ratio $r \equiv t_>/t_<$. The one-loop contributions to the function $\mathcal{C}(t, t')$ are the following

$$\text{figure } 1(a) = \frac{g^2\mu^{-\varepsilon}}{(8\pi)^{d/2}} r^{-2} (\mu^2 \lambda t_<)^{\varepsilon/2} \left[\frac{3}{2}\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\ln r - \frac{1}{5}\frac{1}{r} - \frac{21}{80} - \frac{1}{8}r - \frac{3}{16}r^2 - \frac{3}{8}r^3 - \frac{3}{8}(r^4-1)\ln(1-r^{-1}) + \frac{1}{2}\ln r \ln(1-r^{-1}) + \frac{1}{4}\ln^2 r - \frac{1}{2}\mathrm{Li}_2(1-r^{-1}) \right] \tag{A.1}$$

$$\text{figure } 1(b) = \frac{g^2\mu^{-\varepsilon}}{4(8\pi)^{d/2}} r^{-2} (\mu^2 \lambda t_<)^{\varepsilon/2} \left[\frac{551}{150} - \frac{\pi^2}{10} - \frac{1}{3}\frac{1}{r} - \frac{169}{150}r + \frac{3}{100}r^2 + \frac{129}{50}r^3 + \ln(1-r^{-1}) \left(\frac{21}{50} + 2r - 2r^2 - 2r^3 + \frac{79}{50}r^4 + \frac{2}{25}r^5 - \frac{2}{25}\frac{1}{r} \right) + \ln(1+r^{-1}) \left(-\frac{23}{25} - 2r - 2r^2 - 2r^3 - \frac{23}{25}r^4 + \frac{2}{25}r^5 + \frac{2}{25}\frac{1}{r} \right) + \frac{3}{10}(r^4-1)\mathrm{Li}_2(r^{-2}) + \frac{6}{5}\mathrm{Li}_2(r^{-1}) \right] \tag{A.2}$$

$$\text{figures } 1(c) \text{ and } (d) = \frac{g^2\mu^{-\varepsilon}}{(8\pi)^{d/2}} r^{-2} (\mu^2 \lambda t_<)^{\varepsilon/2} \left[-\frac{1}{\varepsilon} + \frac{2}{\varepsilon}\ln r - \frac{43}{120} + \frac{3}{5}\frac{1}{r} + \frac{1}{6}r + \frac{1}{4}r^2 + \frac{1}{2}r^3 - \ln(1-r^{-1}) \left(\frac{5}{2} - 2r - \frac{1}{2}r^4 \right) - \frac{1}{2}\ln^2 r - \frac{3}{2}\ln r - 2\ln r \ln(1-r^{-1}) - 2\mathrm{Li}_2(1-r) \right] \tag{A.3}$$

$$\text{figures } 1(e) \text{ and } (f) = \frac{g^2\mu^{-\varepsilon}}{(8\pi)^{d/2}} r^{-2} (\mu^2 \lambda t_<)^{\varepsilon/2} \left[-\frac{3}{\varepsilon}\ln r + \frac{9}{4}\ln r - \frac{3}{4}\ln^2 r \right]. \tag{A.4}$$

References

[1] Majumdar S N and Sire C 1996 *Phys. Rev. Lett.* **77** 1420
[2] Janssen H K 1981 *Z. Phys.* B **42** 151
[3] Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
[4] Grassberger P and Sundermeyer K 1978 *Phys. Lett.* **77B** 220
[5] Oerding K, Cornell S J and Bray A J 1997 *Phys. Rev.* E **56** R25
[6] Majumdar S N, Bray A J, Cornell S J and Sire C 1996 *Phys. Rev. Lett.* **77** 3704
[7] Zheng B 1998 *Int. J. Mod. Phys.* B **12** 1419
[8] Bronzan J B and Dash J W 1974 *Phys. Rev.* D **10** 4208
[9] Cardy J 1990 *Phys. Rev. Lett.* **65** 1443
[10] Eisenriegler E, Krech M and Dietrich S 1993 *Phys. Rev. Lett.* **70** 619
[11] Hakim V and Zeitak R unpublished
[12] Hinrichsen H and Koduvely H M 1998 *Eur. Phys. J.* B to appear
[13] Lauritsen K B, Sneppen K, Markošová M and Jensen M H 1998 *Physica* **247A** 1